Fractals from Truchet tilings

H. A. Verrill
www.mathamaze.co.uk
e-mail: hverrill@gmail.com

Abstract

This article concerns a method of producing fractal and space filling curves from certain Truchet tilings. Figure 1 is an example of the kind of work produced. This work illustrates an interplay and interaction between mathematics and art.

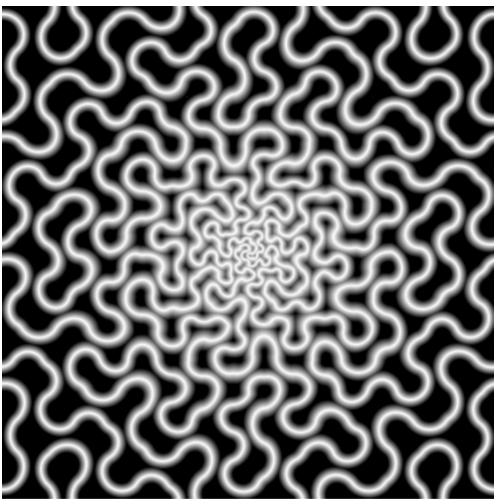

Figure 1: Truchet curve fractally iterated.

The method starts with an array of tiles, each with quarter circles at two opposite corners, as in Figure 1.

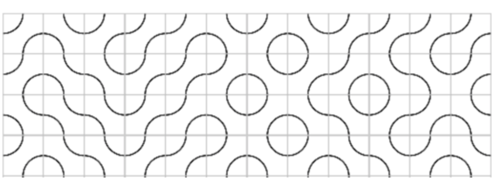

Figure 2: Classic Truchet tiling.

A hinged tiling procedure, depicted in Figure 2, results in a diagonally arranged array of tiles. In the figure, the first step is to add the hinges and rotate, which can be done in two ways. We rotate and scale the tiles, until the spaces between are squares which can be filled with tiles.

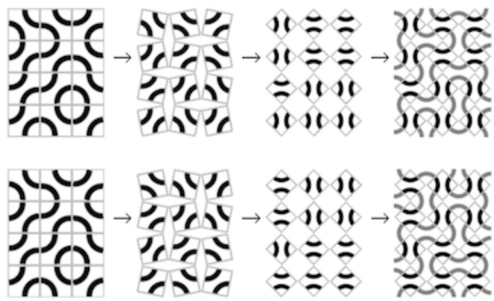

Figure 3: Hinged Truchet tiling.

The process is repeated as often as desired, with a binary choice at each step. Starting from a single circle, some of the curves produced after up to 7 steps are shown in Figure 4. This figure includes the binary tree, and the decision path on this tree used to determine the curve. The resulting curves, which include the fractal dragon curve, can be described as Lindenmayer system space filling fractals.

To smoothly pass from one form to another, instead of considering the iterations as discrete steps, the hinged procedure allows us to consider the step as a continuous progression. The spaces between the initial tiles can be filled from the start with arc segments. Thus, we can continuously traverse a tree of hinged Truchet tilings.

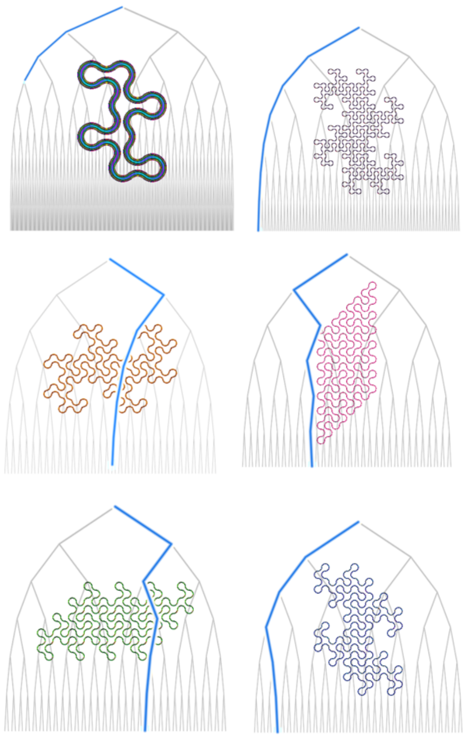

Figure 4: Trouchet curves produced after 3, 7, and 6 iterations of the procedure described, according to the shown path.

We can continuously vary the number of iterations at different parts of the image. In Figure 1, the distance from the centre determines the numbers of iterations. In Figures 5 and 6, the number of iterations is determined by Perlin noise. The method can also generate tessellations (Figure 7). Most of these Figures are frames from Webgl animations, which continuously transform from one curve to another.

The main content of this article is showing how to use a hinged tiling to generate fractal curves from Trouchet tilings. This article can also be seen as a gateway to the study of certain fractals and L-systems, and an example of mathematical exposition through art.

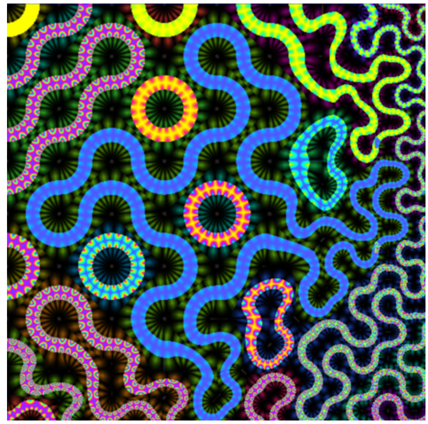

Figure 5: Coloured Trouchet curves.

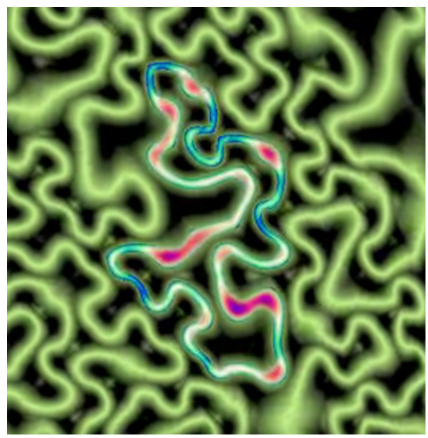

Figure 6: Noisy Trouchet curves.

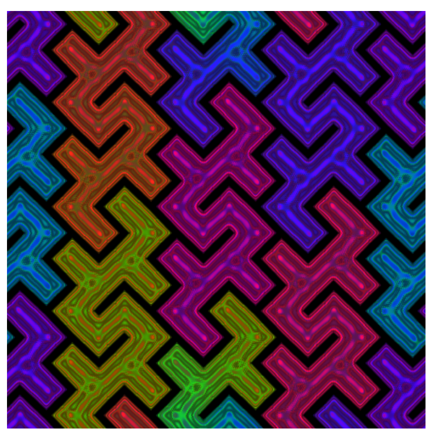

Figure 7: Trouchet tessellation.

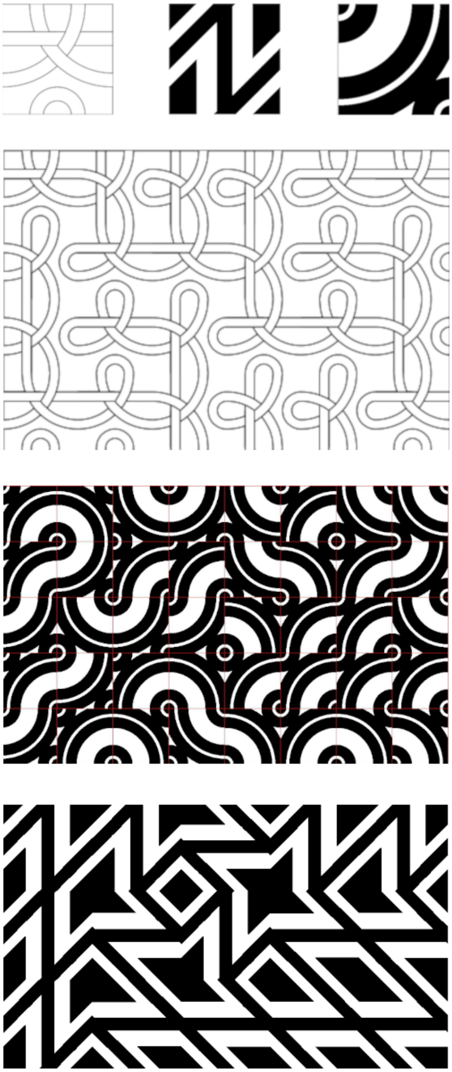

Figure 8: Examples of Trouchet tilings.

1. Truchet tilings

A Truchet tiling, named after Sébastien Truchet [1] is a repetition of a single square tile in a regular array, with different rotations. Figure 8 shows a few examples. Not all tiles will be suitable for the fractal method presented here. A popular tile is Cyril Smith's quarter circles motif [2], as in Figure 1, which I will refer to as Smith's tile. This is the design I will concentrate on. Trouchet tilings are used in generative art and in computer science, e.g., [3], [4].

2. Space filling curves

The path obtained from a Trouchet tiling as in Figure 1 might be referred to as a space filling curve, e.g., in applications to computer storage. However, although it fills a lot of space, it is not as it stands a *mathematical* space filling curve. In mathematics, a space filling curve has points arbitrarily close to any point in the region covered, such as a unit square. Generally, they are constructed by a limiting process. A famous example is the Hilbert curve, which has many variants [5].

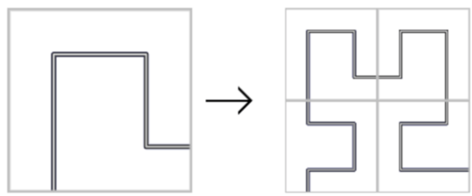

Figure 9: Hilbert curve construction.

Figure 9 shows how one of three tiles is replaced by four smaller versions.

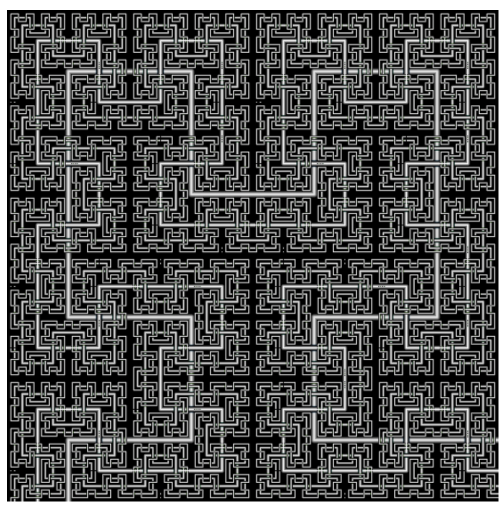

Figure 10: Hilbert curve, 4 iterations

Figure 10 shows several iterations superimposed. The actual Hilbert curve is the space filling curve obtained in the limit as this process is iterated infinitely.

Typical variations as in [5] don't directly apply to Smith's Truchet tiling, because instead of one path through each tile, we have two. Also we start with any random arrangement of the Truchet tiles, which may have several components, not just one. The tiles of the Hilbert construction cannot be put together in any order, whereas we get closed curves however the Smith tiles are arranged. This article describes an iterative method for Smith's Truchet tiling curve, giving a space filling curve in the limit. I describe more of the mathematics of the construction, such as fractal dimension in [6].

3. Hinged tilings

The iterative procedure used here to fill space with a Truchet curve can be thought of as a hinged tiling construction. Originally, I was inspired by considering the upper left tile in Figure 12. The tiling produced (black 5x5 tiling) looks like a superposition of two of the quarter circle tilings of Figure 1. This is clearer when the alternate tiles are coloured red and blue. When the blue tiles are replaced with blank tiles, we see that the resulting tiling can also be obtained by a tiling of the Smith tiles.

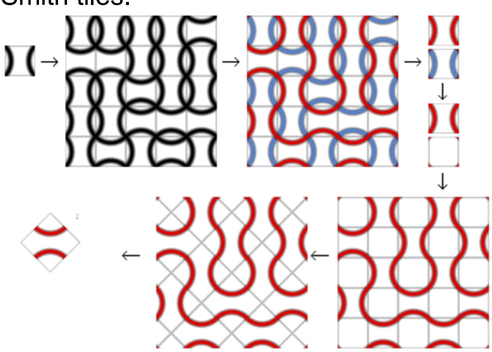

Figure 12: Relationship between two tiles.

To better understand the relationship between the two sets of tiles, let's superimpose the two, and make another tiling, as in Figure 13. I have taken the tiles on lower right Figure 12, tiled alternately, and superimposed this with the original Smith tile, coloured in blue.

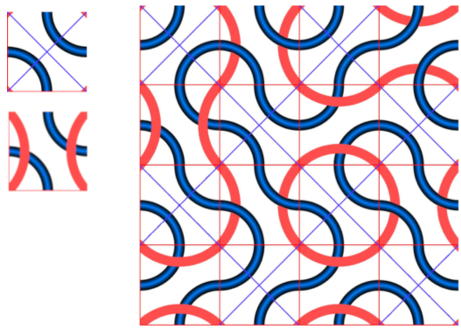

Figure 13: Superposition of tilings.

These tiles are outlined in red, but if we put in diagonal blue lines, we see the red curves become Smith tiles with blue border. Cutting this design into the blue bordered tiles, up to rotation, we have 10 possible tiles, as in Figure 14.

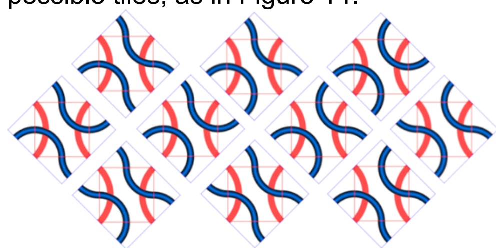

Figure 14: A set of Wang tiles

This is an example of a set of Wang tiles, a set of several tiles, which are only allowed to align in certain ways. In this case, we want edges to match up and paths to continue from tile to tile. If we take only the far right and far left tiles, we can make an almost Truchet tiling, by including the left tile only in odd squares and the right tile only in even squares, where odd and even refers to the parity of the sum of coordinates when tiles are labelled with consecutive integer coordinates. For better deformation properties, we take the right tile and its mirror image. The tile edges always match up, however rotated, provided the two tiles are alternated. Now we have a process for going from one Truchet tiling to another scaled down by a factor of $\sqrt{2}$. Once we've done it once, we can continue, as in Figure 15, which has three sizes of the original Smith tile.

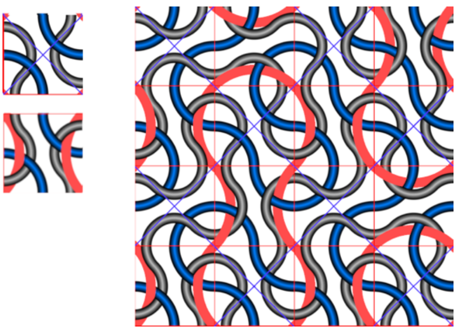

Figure 15: Three generations of tiles, superimposed. Created with the two tiles on the left.

Considering the top left tile in Figure 15, we see that the blue curves can be continuously transformed to the grey curves, as in Figure 16. I have added a grey square background to emphasize that the contents of this square is always the Smith tile. Applying this transformation to each tile, we continuously pass from one size of Truchet tiling to the next. As mentioned previously we need to to mirror alternate tiles, so the intermediate stage is not quite a Trouchet tiling, but the end result is.

Figure 16: Deformation of Truchet tiles.

Putting these together in Figure 17, and looking at the original grey squares, we see we have a hinged tiling.

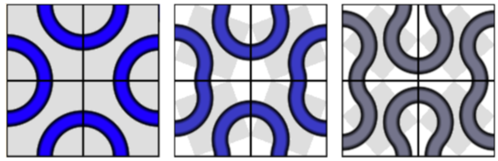

Figure 17: Hinged tiling operation.

In this operation, the original tiles rotate alternately clockwise and anticlockwise, and the new added tiles have alternating rotation. Figure 18 shows how the original tiles have been rotated, and in which orientation the new tiles are added. There are two possible ways to transform. For clarity, the squares which rotate clockwise are pink, and the others are blue.

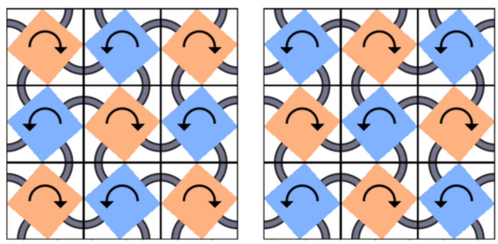

Figure 18: The two ways to transform.

The difference between the iterative step here and other common iterations, such as for the Hilbert curve in Figure 9, is that whereas that procedure divides a square tile into four smaller tiles, replacing each quarter by a scaled and rotated version of the bigger tile, this procedure replaces each tile by two tiles, one of which has the same centre as the old tile.

4. L-systems

An L-system, is a sequence of symbols, typically interpreted as instructions for drawing a path, and a set of rules for transforming these symbols [7]. These can be used to produce fractals, such as the fractal dragon curve [8]. This section is about how to describe the Truchet curves and their transformation in this article, as an L-system. Our symbols are L,R,h,v, interpreted as in Table 1.

L: turn left
R: turn right
h: cross horizontal line (do nothing)
v: cross vertical line (do nothing)

Table 1: Symbols describing a path.

The paths in all the tiles are given a fixed in and out direction, so that which sides are in and out depends on their parity. This is shown in Figure 19, left, where odd tiles are grey, even tiles white. If a tile is rotated through 90 degrees, the direction of flow on the path in that tile reverses.

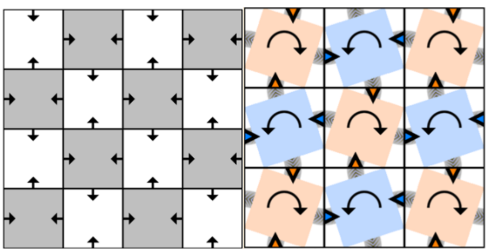

Figure 19: Left: Indication of flow direction on Smith's Truchet tiling. Right: Arrows appearing in horizontal and vertical gaps in the hinged tiling

Figure 20 shows a labelling with L,R,h,v of the paths in a Smith Truchet tiling. The path has arrows to indicate the direction of flow, as in Table 1. For example, the circles have label "RvRhRvRh", or "LvLhLvLh"; the figure of 8 in the upper left is described by "RhRvLhRvRhRvLhRv. Now we give the rules for the L-system. As in Figures 3 and 18, there are two possible operations, which we write in symbols in Table 2. That this is correct can be seen by considering arrows added to the paths in Figure 18 so we can see how the horizontal and vertical lines are replaced with paths in new squares.

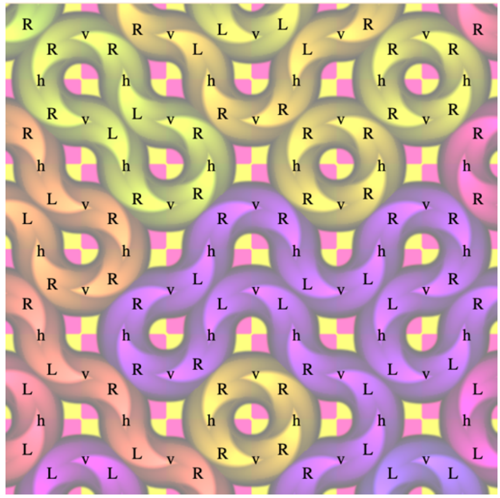

Figure 20: Labelled Trouchet paths.

Operation 1:	Operation 2:
L → L	L → L
R → R	R → R
h → hLv	h → hRv
v → hRv	v → hLv

Table 2: Trouchet L-system operations.

This is shown in Figure 19, right, where the tiles are not turned to the full amount (45 degrees), so that it's clearer which new tiles come from horizontal divisions, and which from vertical divisions. For simplicity, Figure 19 right only includes operation 1. Looking at this figure, it can be observed that all the added curves in the horizontal gaps, marked by orange arrows, are turning right, and all the added curves in the vertical gaps, marked by blue arrows, are turning left. Comparing with the description of the Fractal dragon curve in [8], it can be seen this produces the same structure.

5. Square space filling curve

Any sequence of operations 1 and 2 can be applied. These are depicted as choices in a binary tree in Figure 4, which shows the operations applied to an initial circle. The fractal dragon is achieved by repeated application of operation 1. Different sequences, result in other curves, also studied in [8]. Furthest from the fractal dragon is the curve produced by alternating the two operations. This produces a straight edged region, as in middle right of Figure 4. We can fill a square rather than a diamond shape by instead of a circle, starting with a single quarter circle arc, producing a triangle, and take two of these, as in Figure 21.

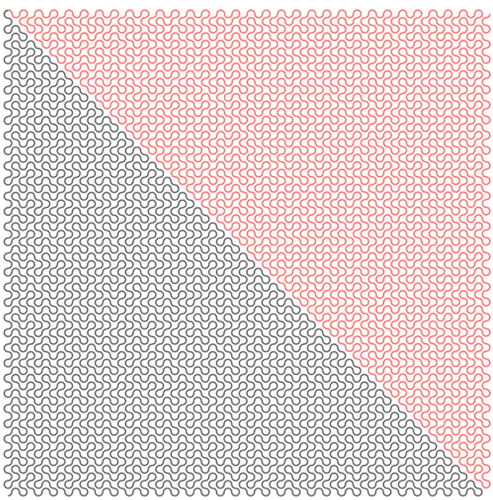

Figure 21: Space filling curve obtained by alternating operations 1 and 2.

6. Tessellations

By construction, since we start with a tessellating tiling, the operations applied always produce tessellations, so we a priori know that the shapes produced tessellate. The shapes in Figure 4 were all produced starting from a single circle, but we can start the process with other starting arrangements of tiles to get different tessellations. We can change from Smith's tile to some other design, for example, a division of the square by diagonal lines. Figure 22 gives examples of such tessellations. Figure 23 also starts from a diagonal line tiling, and superimposes successive iterations, applied to a random arrangement of tiles.

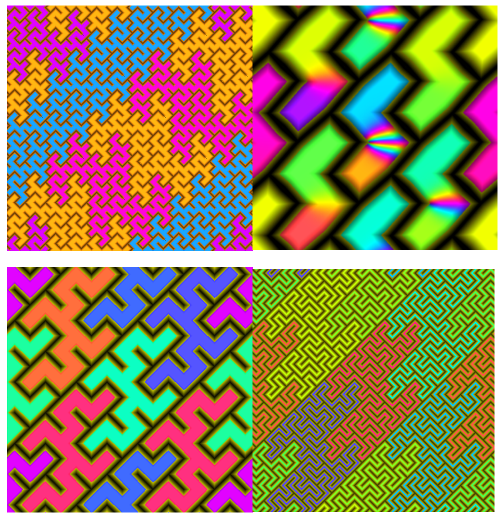

Figure 22: Tessellating tiles.

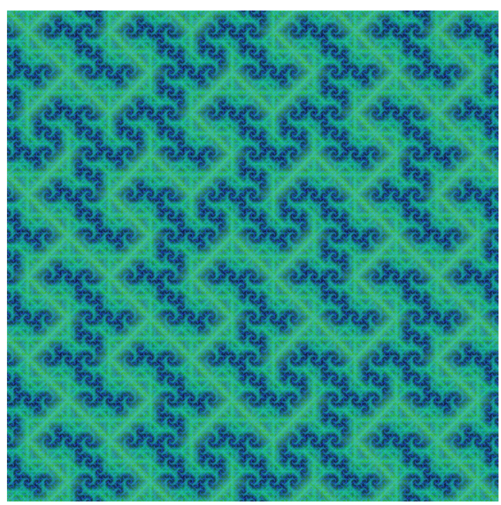

Figure 23: Superposition of iterations.

7. Continuous deformation

Most of the images in this article were generated by the programs available at [9]. In programming these graphics in Webgl, the iteration step is achieved smoothly by a rotating and scaling transformation depending on a parameter t. Choosing t to depend on the pixel coordinates results in images such as Figures 1, 5, 6, 24. This procedure can be applied to any fractal curve where the iteration can be described pointwise. For example, Figure 25 shows an application to the Hilbert curve, from another Webgl program.

Figure 24: Number of Iterations of operations depends on location in image.

References

[1] Truchet, S. 1704. “Mémoire sur les combinaisons”, Mémoires de l'Académie Royale des Sciences, pages 363-372.

[2] Smith, C. S., *"The tiling patterns of Sebastian Truchet and the topology of structural hierarchy"*, Leonardo, ***20*** (4): 373–385, 1987.

[3] Carlson, C. “Multi-scale Truchet patterns.” Proceedings of Bridges 2018: Mathematics, Art, Music, Architecture, Education, Culture, pages 39–44, 2018.

[4] Browne, C. “Truchet curves and surfaces”, Computers & Graphics, 32(2):268–281, 2008.

[5] Ozkaraca, M. I. “An application of space filling curves to substitution tilings.” PhD thesis, Glasgow 2021.

[6] Verrill, H. 2021 “Hinged Truchet tiling fractals”, arXiv: 2110.01069 [Math DS].

[7] Lindenmayer, A. “Mathematical models for cellular interactions in development ii. simple and branching filaments with two-sided inputs.” Journal of Theoretical Biology, 18(3):300–315, 1968.

[8] Tabachnikov, S. “Dragon curves revisited” Math. Intelligencer, 36(1):13–17, 2014.

[9] Verrill, H. www.mathamaze.co.uk/truchet/

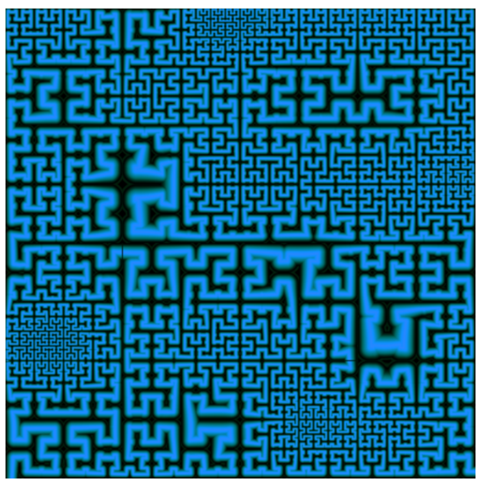

Figure 25: Iteration of the Hilbert curve depending on position on image.