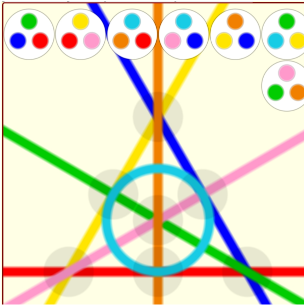
Picturing Automorphisms of the Fano Plane

H. Verrill, Bridges 2025, Eindhoven



Have a go with a Dobble/Fano inspired game

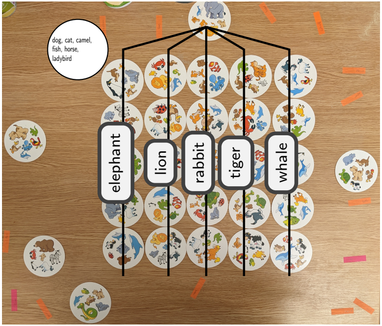
Dobble (Spot It!) and projective planes

Dobble (Spot It!) works because every pair of cards have a common symbol

Dobble (Spot It!) and projective planes

These cards can be arranged so that:

- There are 6 sets of “parallel” lines, meeting at points at “infinity”



lines of the form $\{(\alpha s : t : s) : (t : s) \in \mathbb{P}^1\}$

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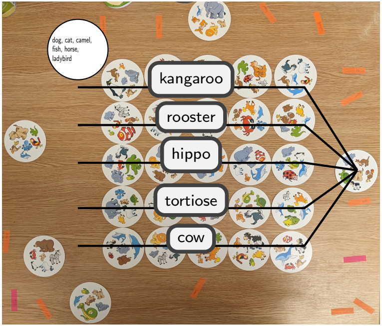

lines of the form $\{(t : \alpha s : s) : (t : s) \in \mathbb{P}^1\}$

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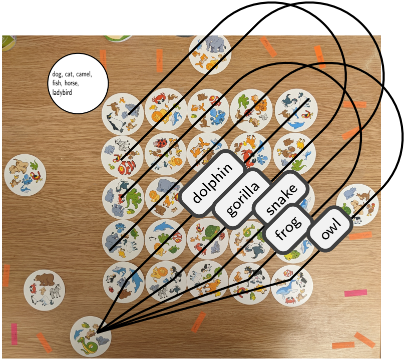

lines of the form $\{(t : t + \alpha s : s) : (t : s) \in \mathbb{P}^1\}$

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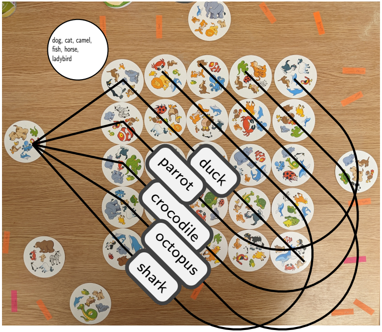

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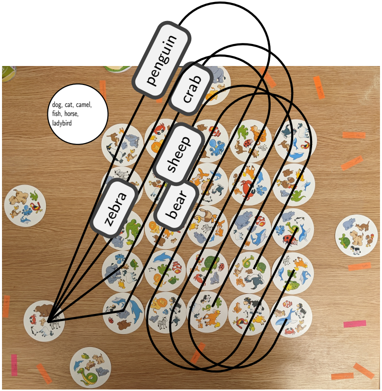

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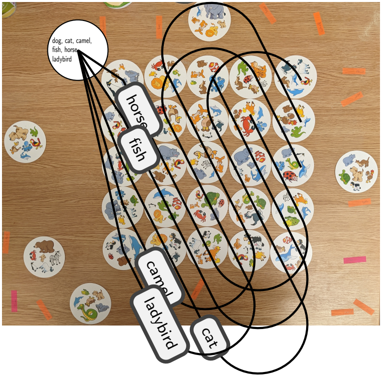

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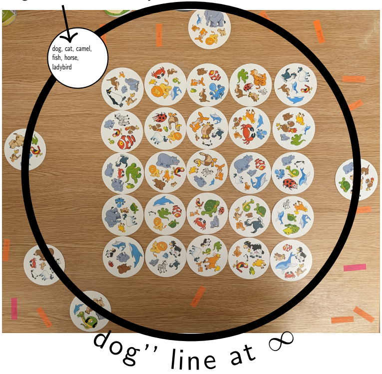
Dobble (Spot It!) and projective planes

These cards can be arranged so that:

- There are 6 sets of “parallel” lines, meeting at points at “infinity”
- And a line at “infinity”

missing card:

dog,cat, camel, fish, ladybird, horse



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Note that these cards could have been laid out in 372000 different ways, with the cards all in the same grid pattern, and with each card on 6 lines cooresponding to its symbols. Not all permututations of lines or points will preserve point/line relationships.

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Exercise:
Lay the cards out in a different way.

Axiomatic Projective plane (Gino Fano 1892)

A finite axomatic projective plane is a set of points and lines such that:

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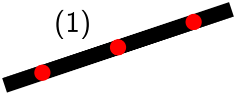

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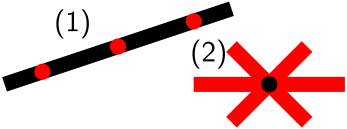
- 

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A finite axomatic projective plane is a set of points and lines such that:

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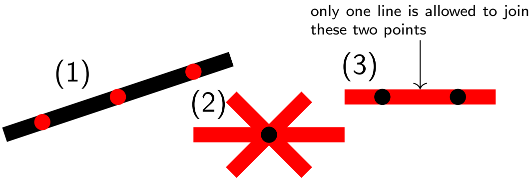
- 

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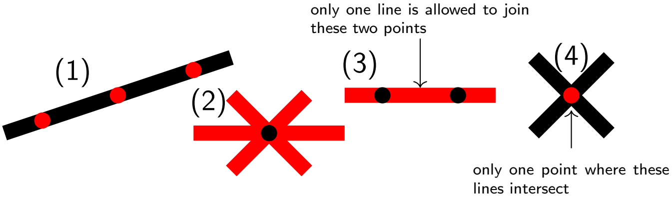

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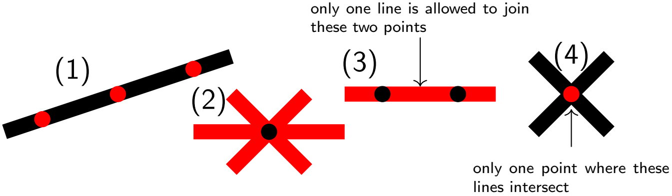
- 

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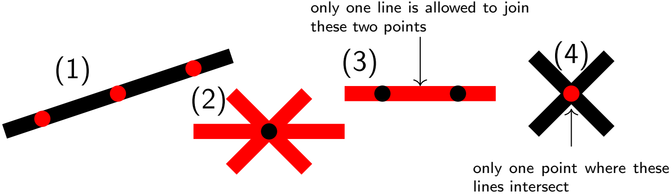
- 

  Exercise: Construct your own examples, or verify given examples.
- If these axioms hold, for some n, there are $n+1$ points on each line, $n+1$ lines through each point, and n^2+n+1 lines and n^2+n+1 points. n is the **order** of the plane.

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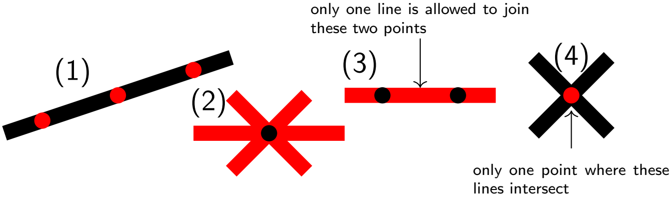
- 

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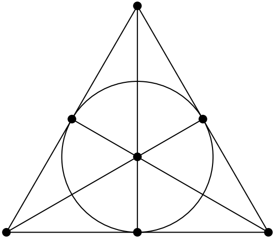
The Fano Plane

The smallest axiomatic projective plane is the **Fano Plane** with order 2:



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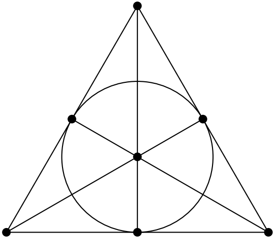

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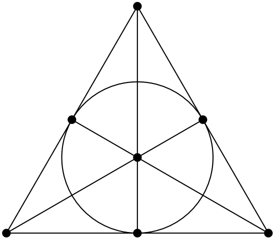

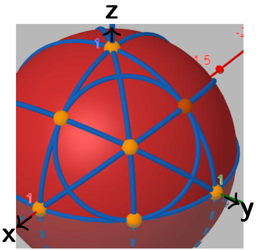

- What's projective about this?
- The vectors $(1:0:0),(0:1:0),(0:0:1),(1:1:0),(1:0:1),(0:1:1),(1:1:1)$ project to these points on a sphere. (... projective geometry...)

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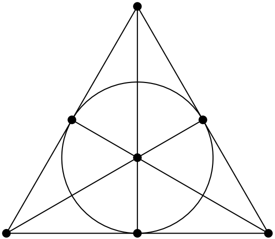

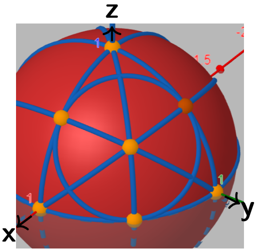

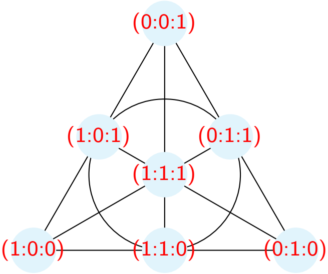

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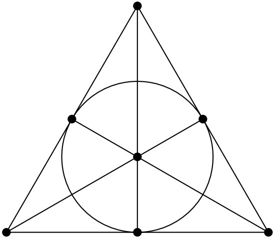

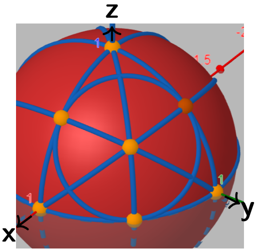

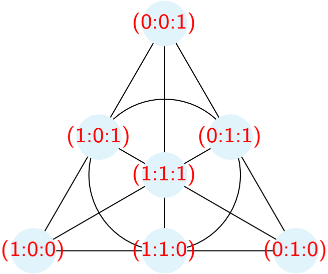

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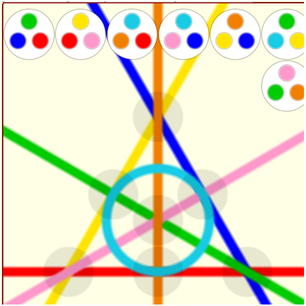
- You can use these coordinates to describe these points.
- We only need to think of these modulo 2 for the Fano Plane.

Now let's invent a game

Place the cards so that each card lies on lines with the same colour as the dots on the card

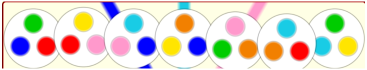
Fano's plane



https://www.mathamaze.co.uk/circles/fano/

How to arrange cards on lines

Given a set of “mini Dobble” cards,

- 

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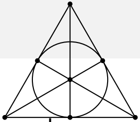

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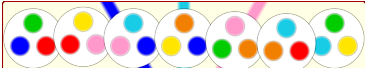
- 

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  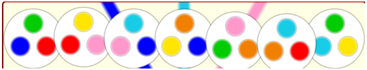

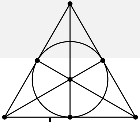

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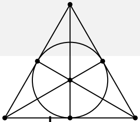

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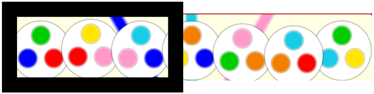

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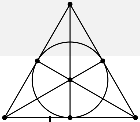

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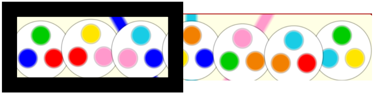
- 

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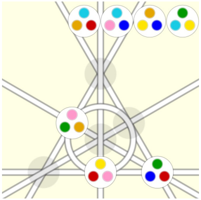

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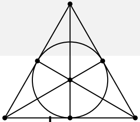

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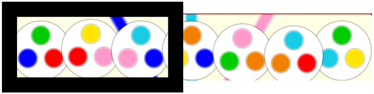

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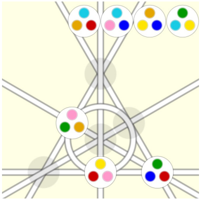

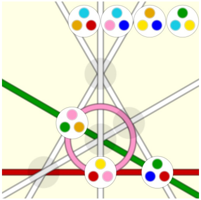

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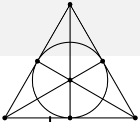

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  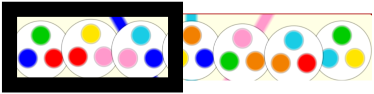

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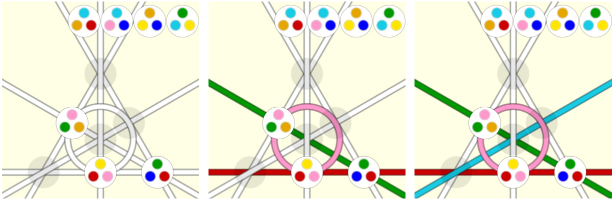

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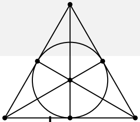

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  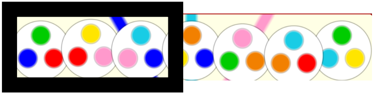

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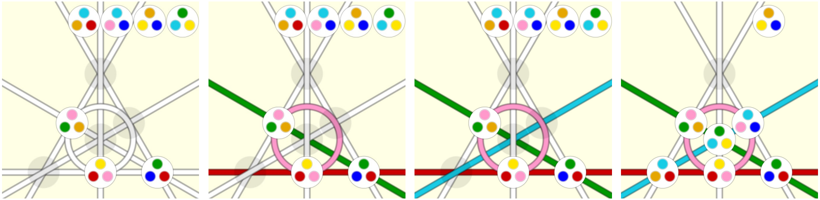
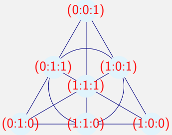


Automorphisms of the Fano Plane



Let's call the Fano plane...

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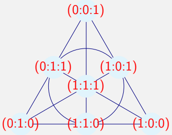

Let's call the Fano plane... $\mathbb{P}^2(\mathbb{F}_2)$ for short

Automorphisms of the Fano Plane $\mathbb{P}^2(\mathbb{F}_2)$

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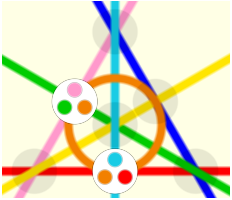
- An automorphism of something preserves its structure...

$\rightarrow$



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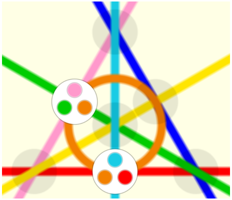

For short, write the map, which is a permutation of points, as:

$$f : \mathbb{P}^2(\mathbb{F}_2) \rightarrow \mathbb{P}^2(\mathbb{F}_2)$$

Structure: if P is a point on a line L, then $f(P)$ is a point on a line $f(L)$.

Automorphisms of the Fano Plane $\mathbb{P}^2(\mathbb{F}_2)$

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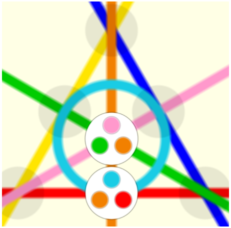
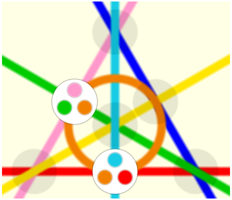


$\to$



For short, write the map, which is a permutation of points, as:

$$f : \mathbb{P}^2(\mathbb{F}_2) \to \mathbb{P}^2(\mathbb{F}_2)$$

Structure: if P is a point on a line L, then $f(P)$ is a point on a line $f(L)$.

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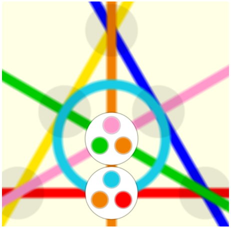

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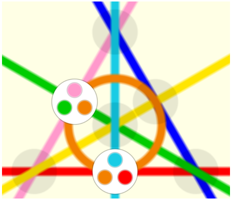

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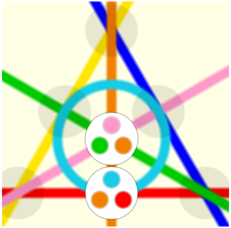

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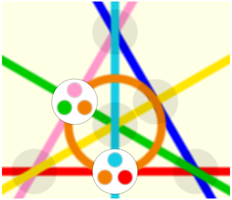

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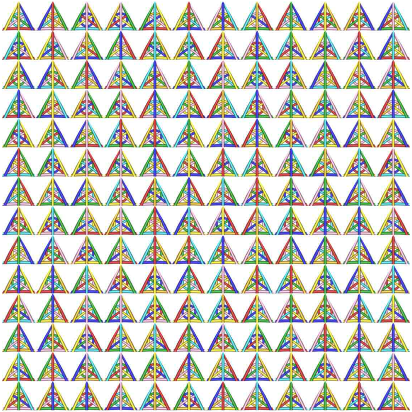

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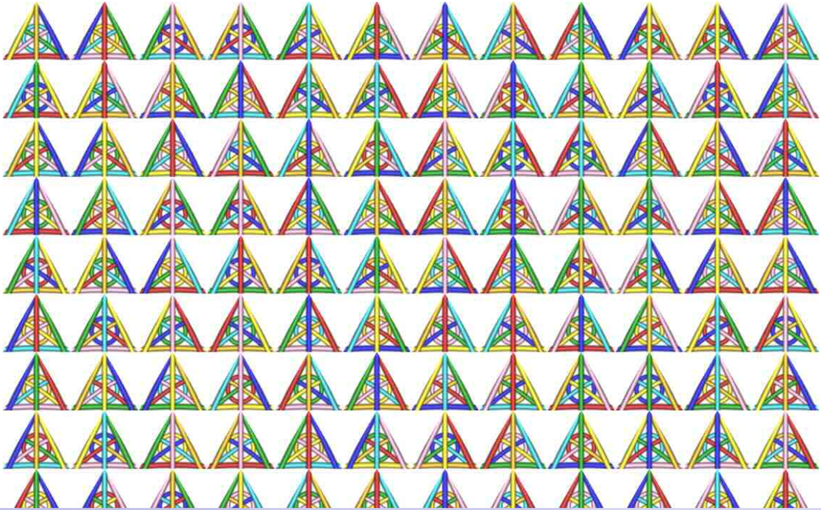

Sequencing the automorphisms

- Let's look at this sequence in time, instead of laid out in space.

  [interlude to look at animation (generator choice 336)]
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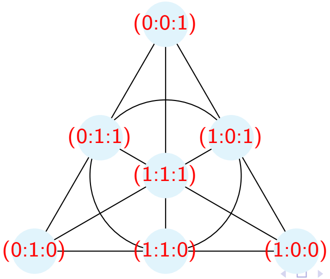
- The sequence based on chosing card for position (1,0,0), then card for position (0,1,0), then card for position (0,0,1) does go through all arrangements, but is not very "ballanced", e.g., card at position (1,0,0) stays there for a long time and gets bored.



The Klein Quartic

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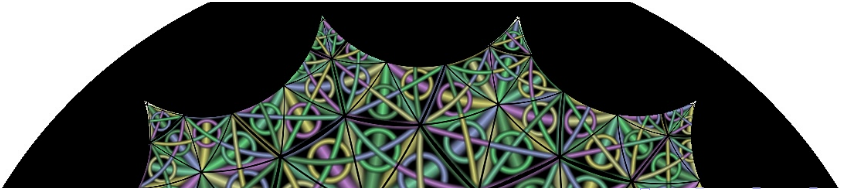

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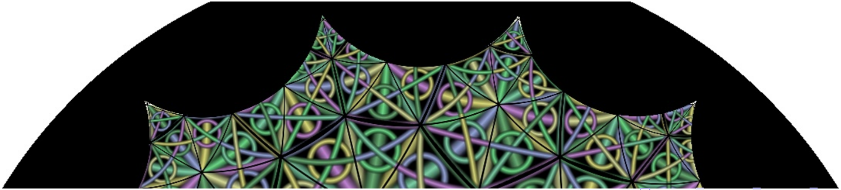

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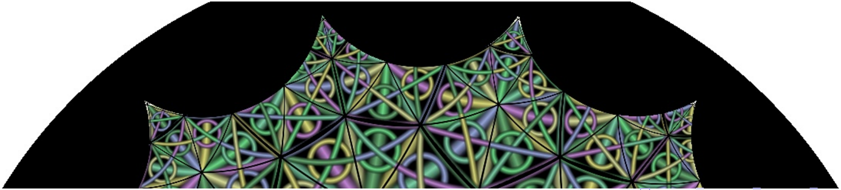

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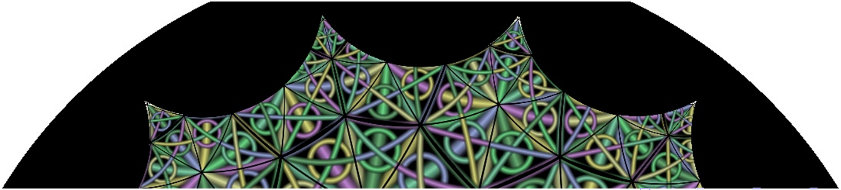

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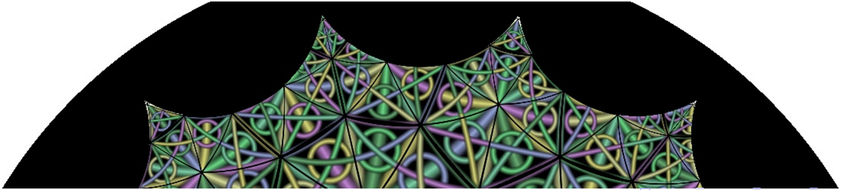
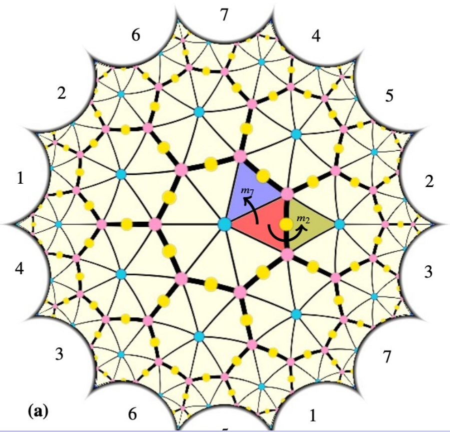
- But what are these objects, and what are automorphims groups?
- We already know about the Fano plane, and we've essentially listed the elements of $PGL(3,2)$, pictorially, so next onto Klein quartic.



The Klein Quartic and its Automorphisms

- We can represent the Klein quartic as a collection of hyperbolic triangles (they want to live in hyperbolic space – you can't tile the plane with regular heptagons...), with edges glued together, according to numbers in the figure...



The **Automorpshisms** are the symmetries of this object.

Any symmetry corresponds to a choice of triangle; the symmetry will map the red triangle to another triangle.

Any symmetry can be acheived by applying a sequence of rotations of order 7 and of order 2, about blue and yellow vertices; m_7 and m_2 in the figure. Note that $(m_2/m_7)^3 = 1$.

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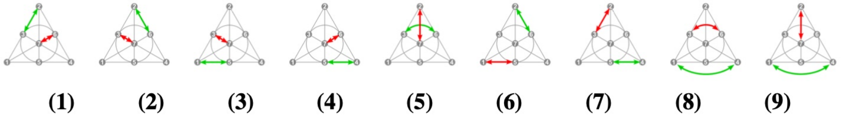

Figure 6: *A selection of Fano automorphisms or order* 2 *[10].*

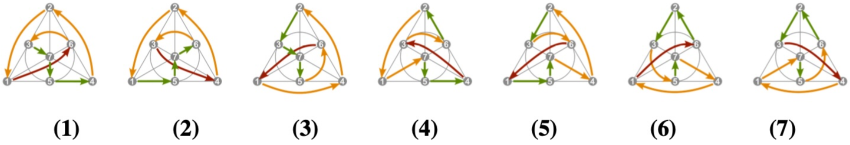

Figure 7: *A selection of Fano automorphisms or order* 7. *[10]*

(from [10] T. Piesk. 3-bit Walsh permutation/cycle shapes, Wikiversity.)

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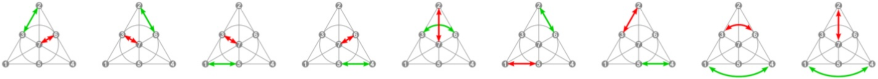

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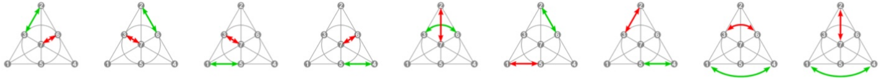

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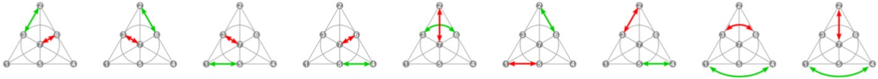

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- Idea for choosing a “nice” way to pass through the automorphsims of the Fano plane:
- choose a pairing between generators; this determines the pairing for the whole of the groups.
- Place the arrangement of the Fano plane on the corresponding triangle of the tiling of the Klein quartic.

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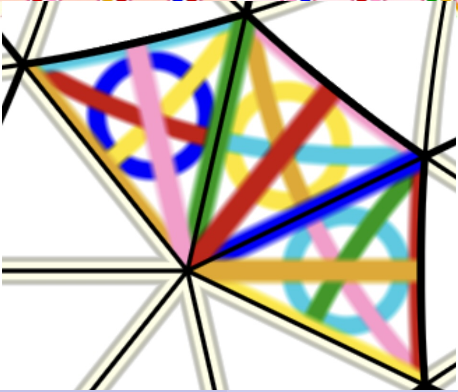
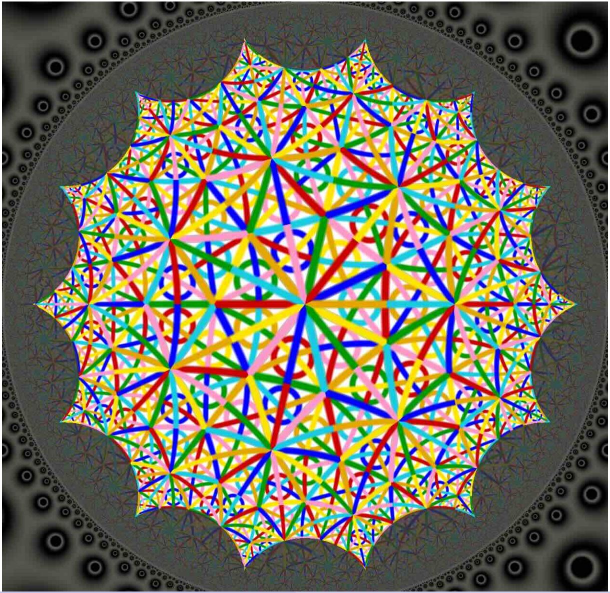


E.g., when passing through an order 7 rotational symmetry from one triangle to the next, the Fano pictures must be related by a permutation that matches the edges, e.g, here are first few applications of a suitable order 7 permutation:

Example isomorphism of automorphism groups



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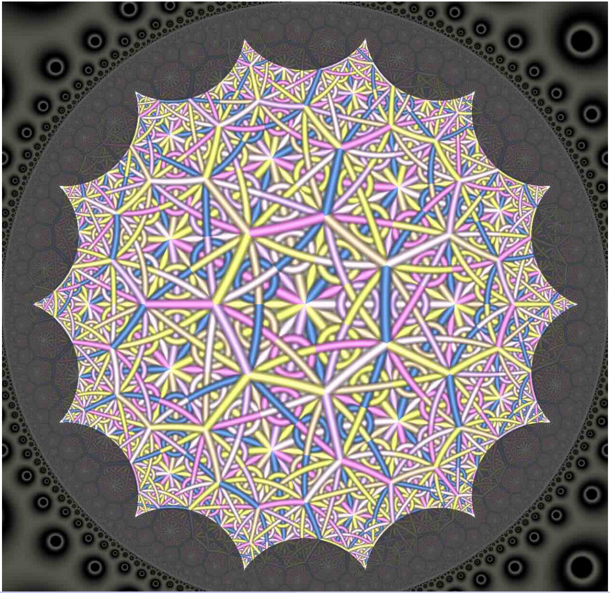
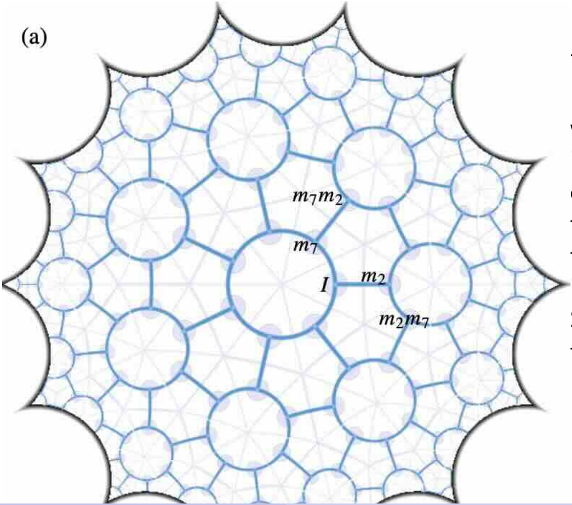


Cayley graph

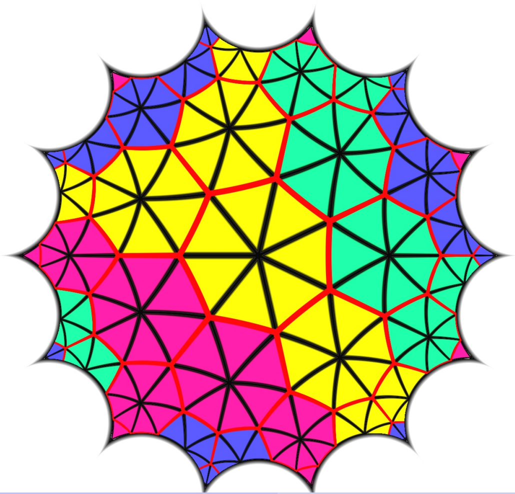
Cayley graph of a group – each vertex corresponds to a group element; edges correspond to certain generators:

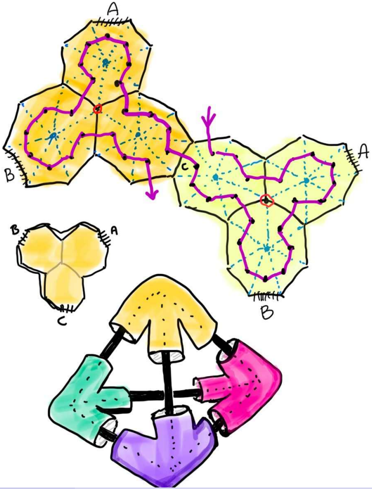


To pass through the automorphisms once each in a "nice" way, where "nice" means limiting to only using the two generators (and maybe inverses of them), means finding a path through this graph, only passing through each vertex once. Such a path is called a Hamiltonian path.

Pants decomposition

To simplify the problem of finding such a path, we can divide up into "pants", and find a path on each pair of pants:





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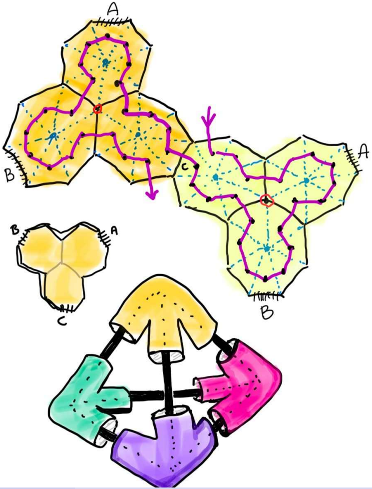
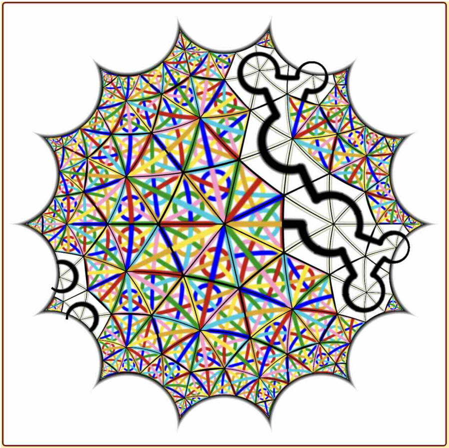
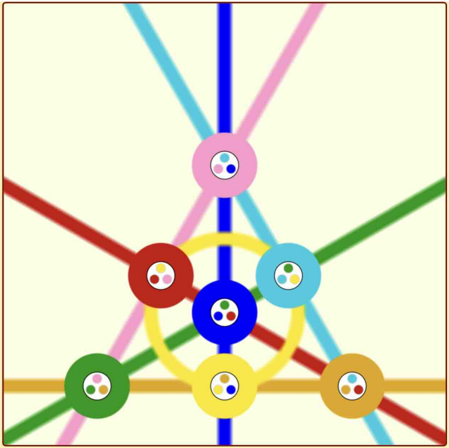


Animation

Finally, putting it altogether, we have an animation where we can see the repeated permutations, passing through all cases.





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