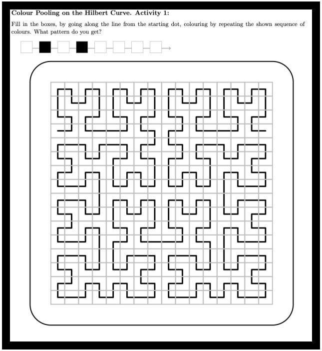
Color Pooling on the Hilbert Curve

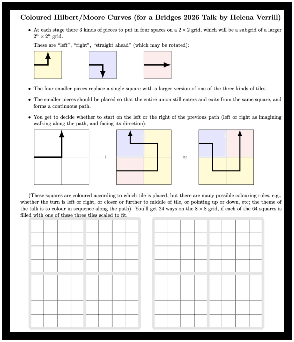
H. Verrill, Bridges 2026, Galway

Before we start: handouts & animation & show&tell



Colour Pooling on the Hilbert Curve. Activity 1:

Fill in the boxes, by going along the line from the starting dot, colouring by repeating the shown sequence of colours. What pattern do you get?



Coloured Hilbert/Moore Curves (for a Bridges 2026 Talk by Helena Verrill)

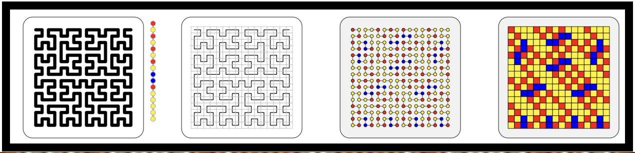
- At each stage there 3 kinds of pieces to put in four spaces on a 2×2 grid, which will be a subgrid of a larger $2^n \times 2^n$ grid.

  These are "left", "right", "straight ahead" (which may be rotated):

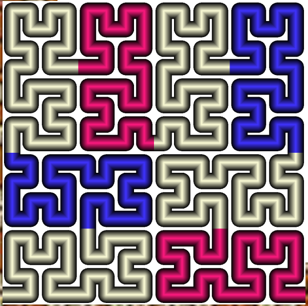
- The four smaller pieces replace a single square with a larger version of one of the three kinds of tiles.
- The smaller pieces should be placed so that the entire union still enters and exits from the same square, and forms a continuous path.
- You get to decide whether to start on the left or the right of the previous path (left or right as imagining walking along the path, and facing its direction).

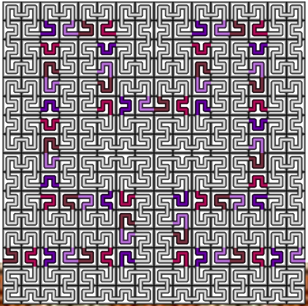
$\longrightarrow$ or

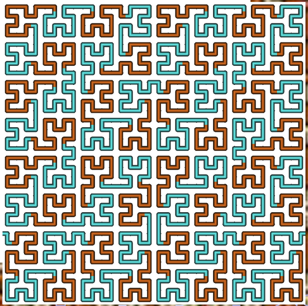
(These squares are coloured according to which tile is placed, but there are many possible colouring rules, e.g., whether the turn is left or right, or closer or further to middle of tile, or pointing up or down, etc; the theme of the talk is to colour in sequence along the path). You'll get 24 ways on the 8×8 grid, if each of the 64 squares is filled with one of these three tiles scaled to fit.



What is colour pooling of the Hilbert curve? with examples







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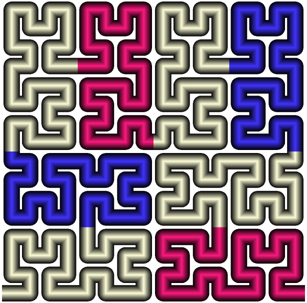

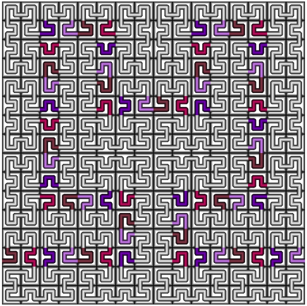

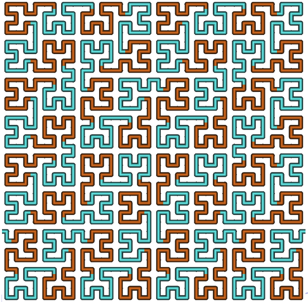

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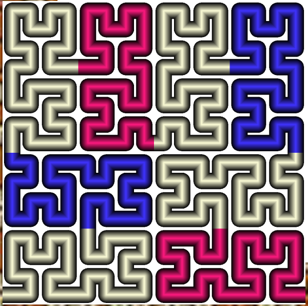
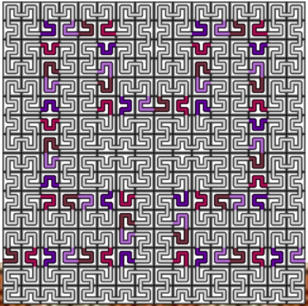
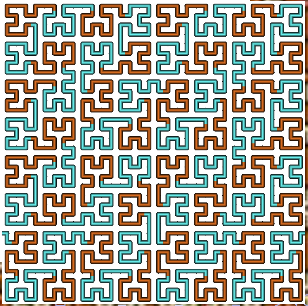

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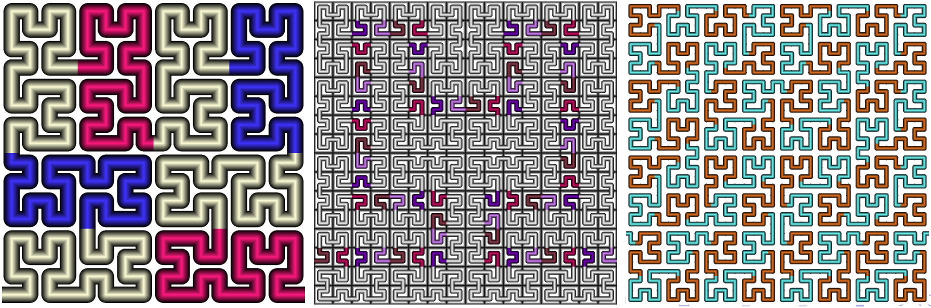
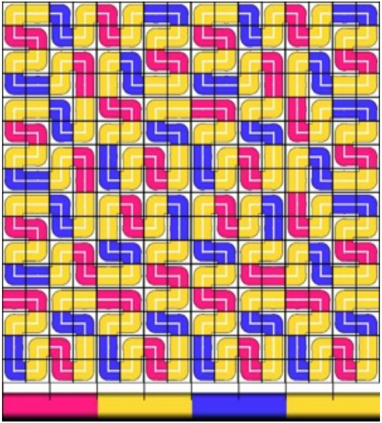
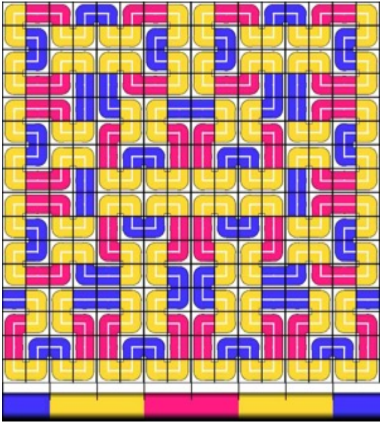
- Then I'll talk about Hilbert curves (maths) (with saying what the Hilbert curve is) (See handout 2)



Colour Pooling - examples and observations





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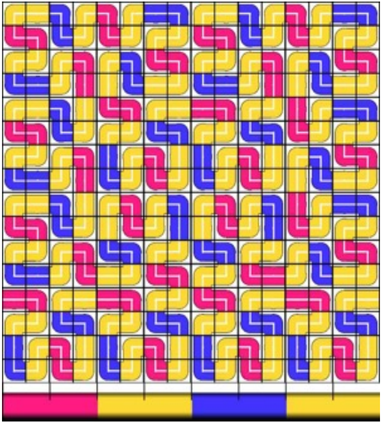

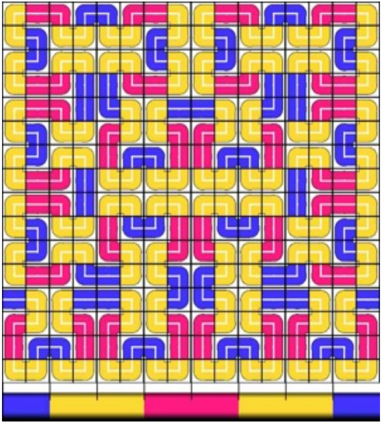

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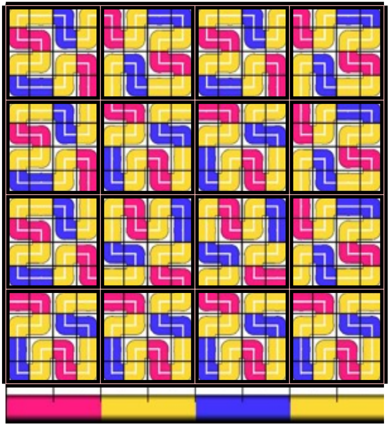

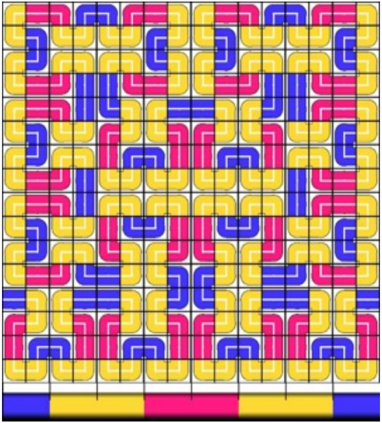

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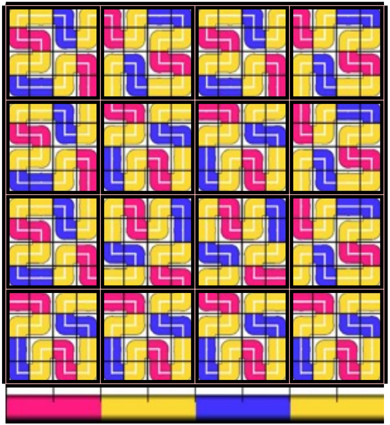
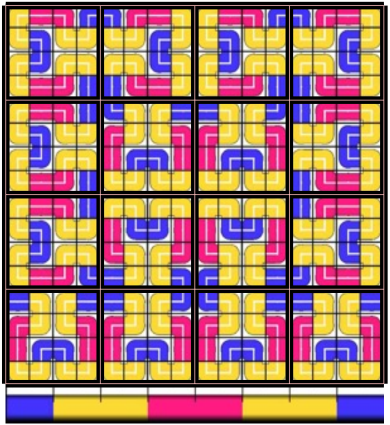
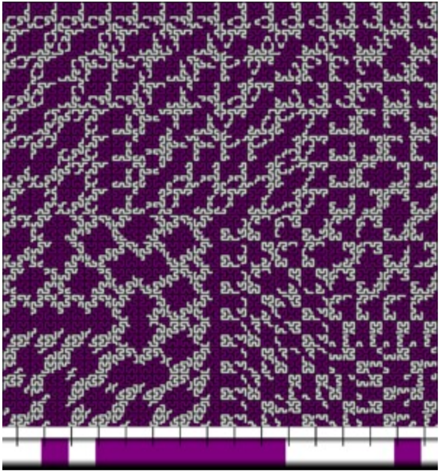


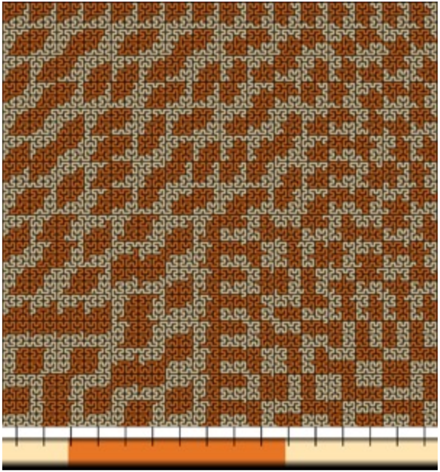


- Stitch length a mulitple of unit path length for regular pattern
- Repeat unit length a multiple of 2^{2n} for repeating blocks
- Same curve, same thread, but offset gives different patterns

Interesting colour pooling effect

- If the repeat unit is a tiny bit longer than 2^n, eg, 256.25, the pattern will gradually misalign, and then realign to a different pattern





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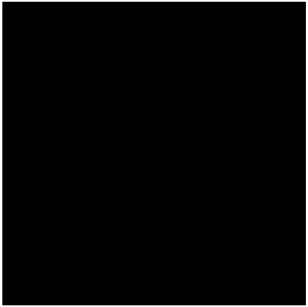

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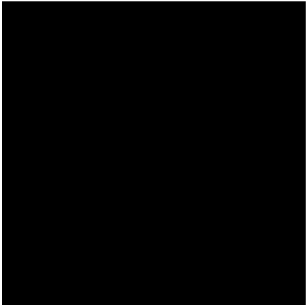

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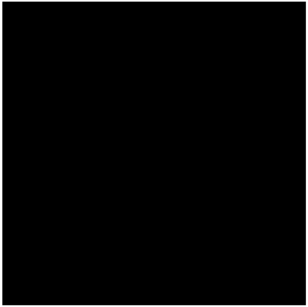
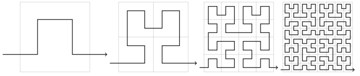


- It's a continuous path through space, that passes through every point of a square... but you can't really travel along it. You'd have to travel infinitely fast to go approximately nowhere.

The Iterates of the Hilbert Curve



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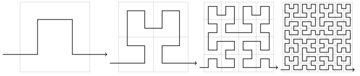

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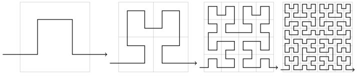

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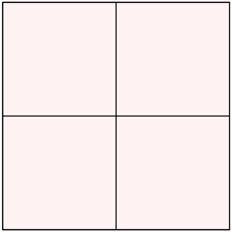
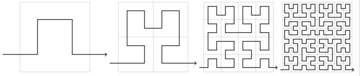
- 



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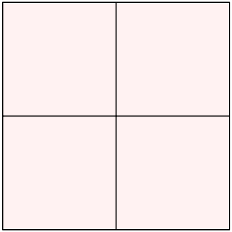
- 

  2×2 grid

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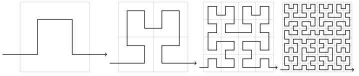

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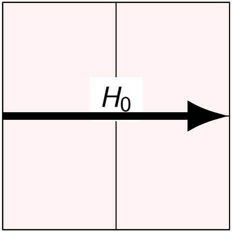
- 

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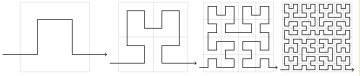

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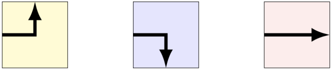
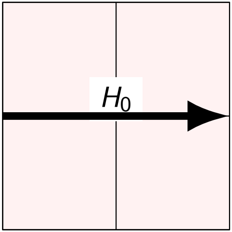


the three possible tiles



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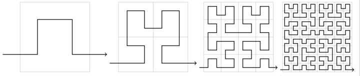

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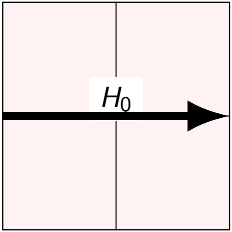

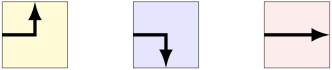

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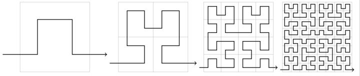

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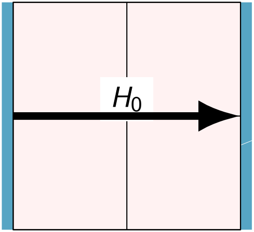
- 

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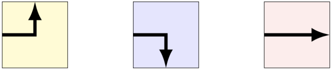

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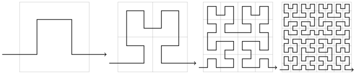

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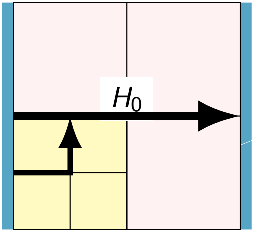
- 

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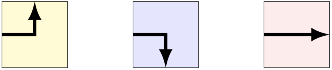

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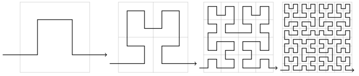

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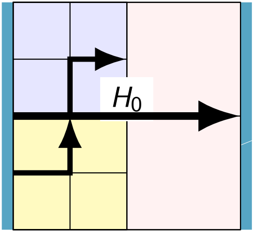

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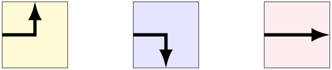

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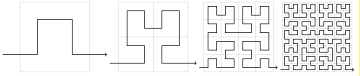

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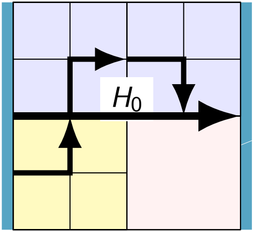
- 

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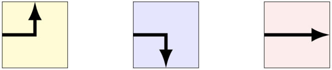

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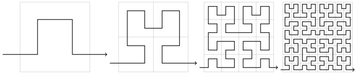

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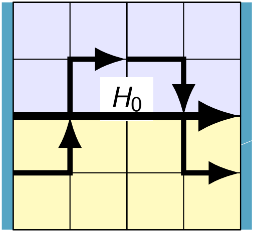
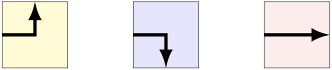
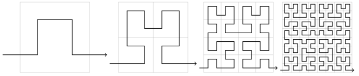
- 

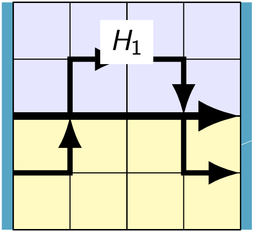
2×2 grid for H_0



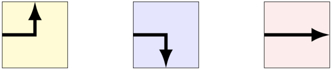
the three possible tiles

The Iterates of the Hilbert Curve



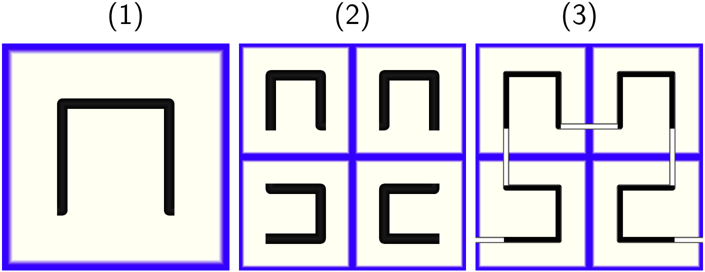
- The Hilbert curve is a limit of a sequence of curves, $H_0, H_1, H_2, \ldots$
- There are several different ways to describe H_n. My favourite:
- H_n is made of lines drawn on a grid of $2^{n+1} \times 2^{n+1}$ squares
- It's made of $2^n \times 2^n$ "left", "right", "straight" tiles,
- At each stage, we replace each tile of H_n by 4 smaller tiles:
- The complete path has to start and end on the same sides.
- 

4×4 grid for H_1



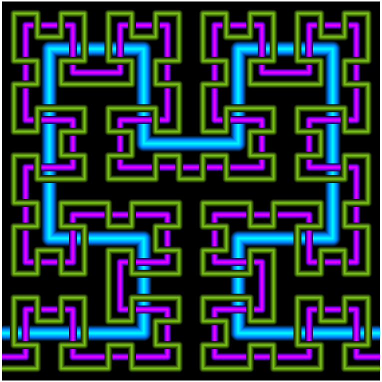
the three possible tiles

Possibly more frequent explanation of Hilbert's iteration rule



- (1) The tile on the left (4 orientations), showing a "bump"
- (2) is replaced by the 4 smaller "bump" tiles in (2) (4 orientations)
- (3) (for continuous paths, three cases to consider; i.e., adding the white bits of paths on right diagram)

Hilbert variations

- For the Hilbert curve, each sucessive path starts to the right of the previous path.
- 

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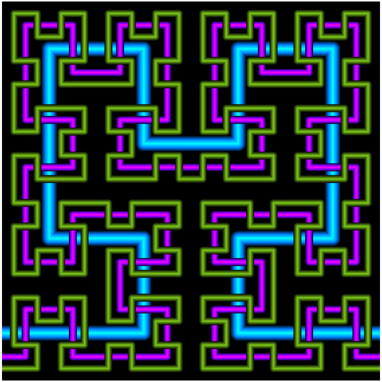
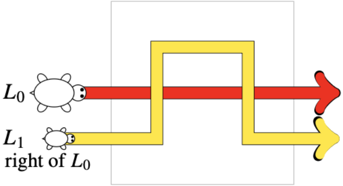
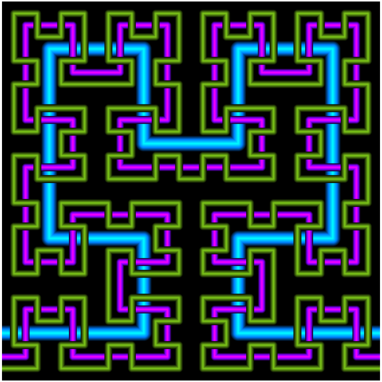
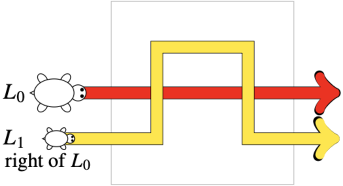
-  

Hilbert variations

- For the Hilbert curve, each sucessive path starts to the right of the previous path. (meaning of “right” on right)
- But we could start each successive curve on the left.
-  

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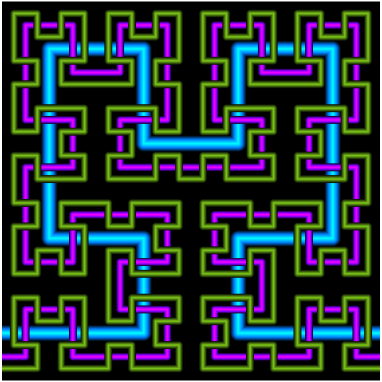
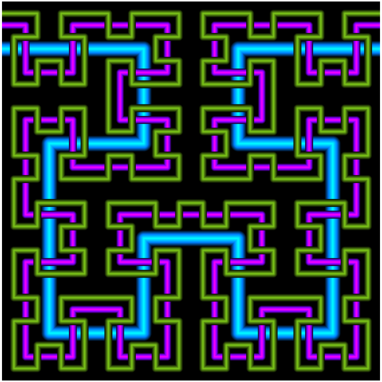
-  

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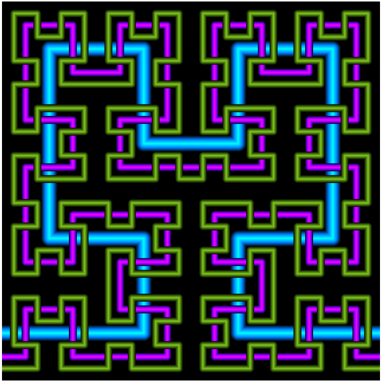
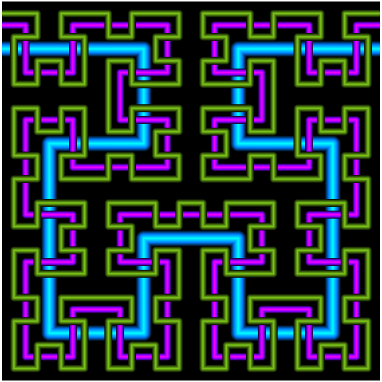
-  
- A reflection...

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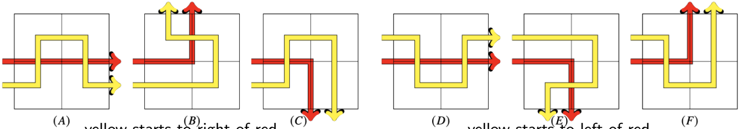

yellow starts to right of red

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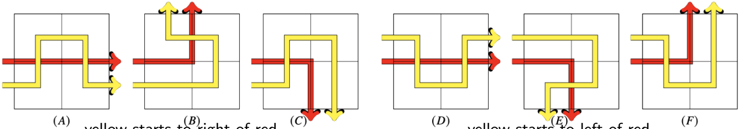

yellow starts to right of red   yellow starts to left of red

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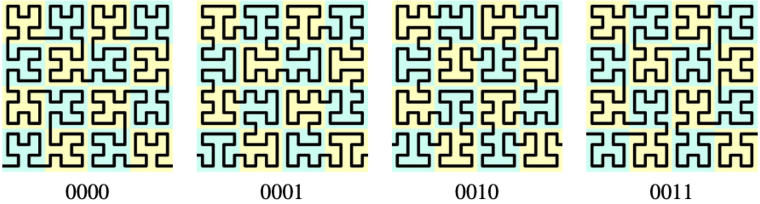

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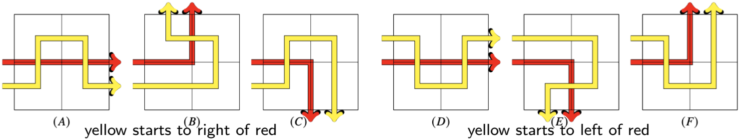

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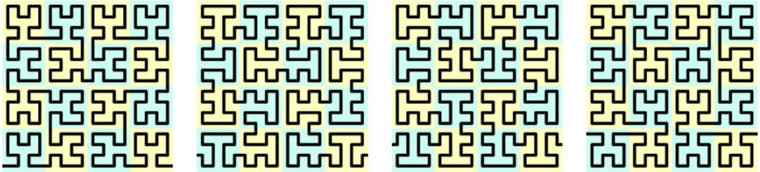
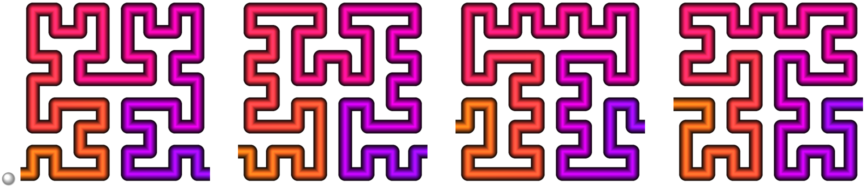


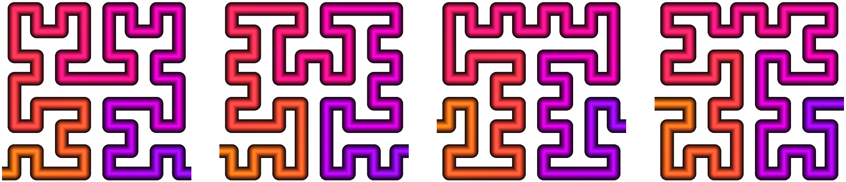
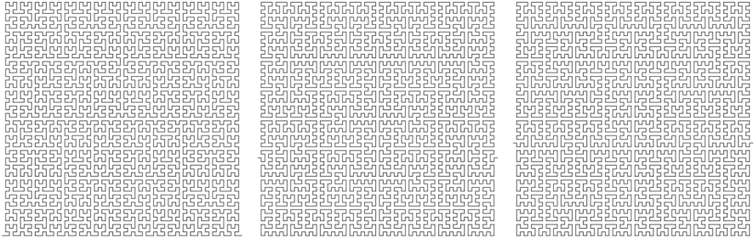
0000 0001 0010 0011

- Michael Bader's book on space-filling curves's $\beta\Omega$ corresponds to 010101 . . . . Zuguang Gu 2025 arXiv paper covers all cases.

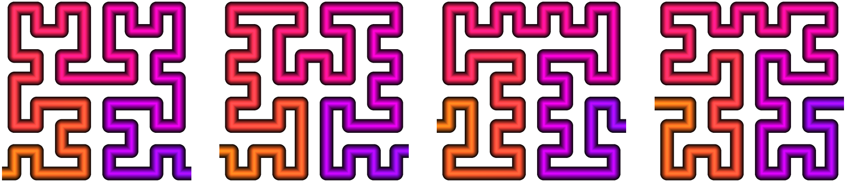
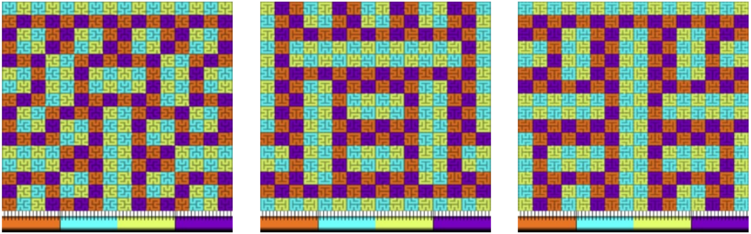
Third iterates



Third iterates and some 6th iterates

- 
- 

Third iterates coloured and some 6th iterates

- 
- 

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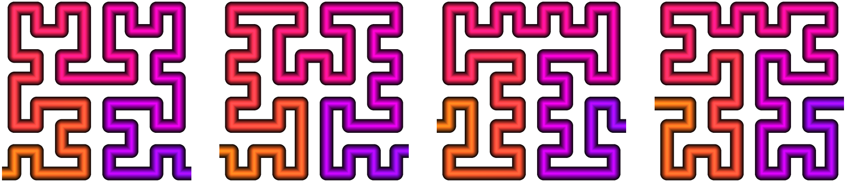
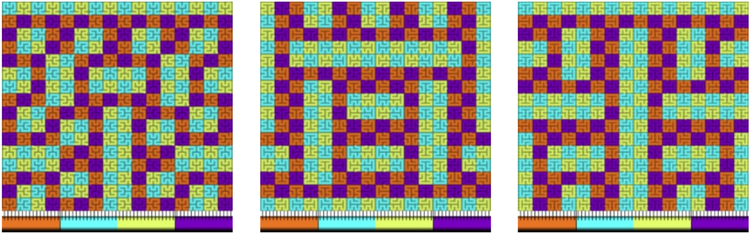
- 
- 
- Different sequences are more easily distinguished when coloured

L-system for Hilbert curve

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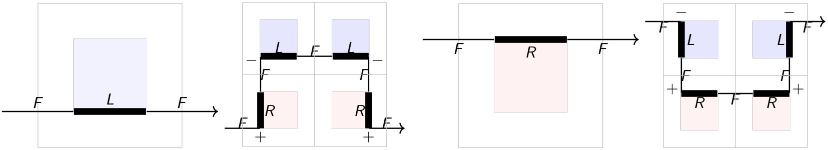
- 

What was a "bud" for one iterate is replaced by a sequence of turns and moves and "buds" for next iterate.

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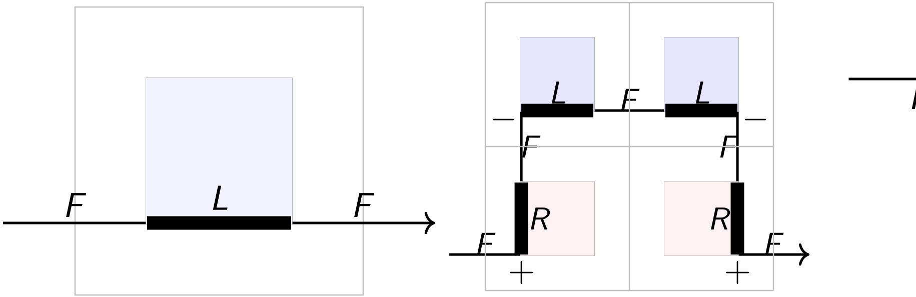
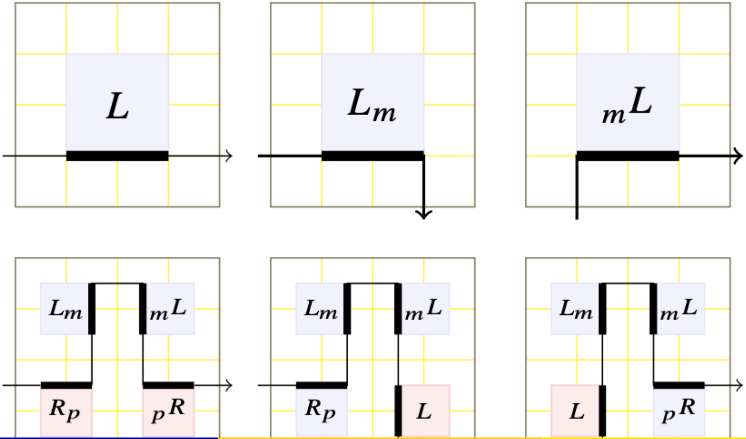


R case looks like a reflection of this

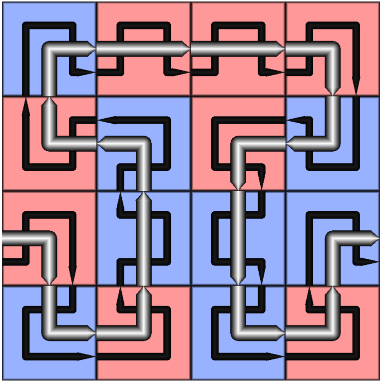
Compatible L-systems (the new maths)

In order to have compatible L-system operations, we have to instead think about whether the path moves closer or further from a central line in each square... We get different symbols (L, ${}_mL$, L_m, R, ${}_pR$, R_p), and rules like $L \mapsto R_p L_m \, {}_mL \, {}_pR$



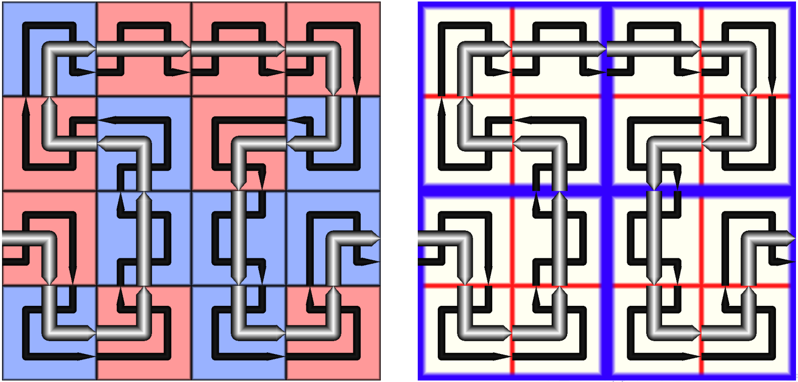
Inspiration for new L-system discovery

- Unfortunately, whether a path starts on left or right of previous is not conserved over the tiling.
  Here on red tiles, the next iteration is right; on blue it is left.



Inspiration for new L-system discovery

- Unfortunately, whether a path starts on left or right of previous is not conserved over the tiling.
  Here on red tiles, the next iteration is right; on blue it is left.
- What **<u>is</u>** conserved is whether the new path is closer or further from the mid-lines (in red) of the tile, at entry point



The art Thanks to Margaret Low (WMG) & Warwicks Engineering Build Space

- Discussion of artistic interpretations:
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- WebGL version
- Sewing machine
- 3D printing
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Thank you!
Questions?