VARIATIONS ON SQUARE TWIST/MOMOTANI BRICK WALL

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One of the simplest origami tessellations to fold is Momotani's brick wall [M].

This could be viewed as a lot of "bricks" put together, or as a lot of square twists put together; i.e. possibly basic units of the tessellation could be thought of as in Figure 1

The crease lines are given in Figure 2, which shows the lines of the creases, but not whether they are mountain or valley folds. By using the same crease pattern, but folding the creases in different directions, many different tessellation patterns can be acheived. Some of these are shown in Figure 3. Each pattern can be continued infinitely, e.g., at the very least by reflection along the edges, or in most cases by translation. Certain cases would better be repeated in other ways. These are fairly small samples; with a bigger sheet of paper more patterns are possible, for example, the piled up "tower" in Figure 3 could be continued as in Figure 4. The herring bone attempt from [V], here in Figure 4 is also easier to see in a larger sheet of paper. It's also possible in a larger sheet of paper to transition between different choices of these tessellations, as for example in Figure 14, where we have "weaving", "bricks" and "square twists" all in the same sheet.

But I will just look at the variations on the crease pattern in Figure 2.

Note that although these are all flat folding tessellation patterns, in the models, I have not squashed the paper flat, so some texture can be seen, but this means that there are extra folds which appear, particularly across the diagonal of the squares. I will ignore these in the diagrams.

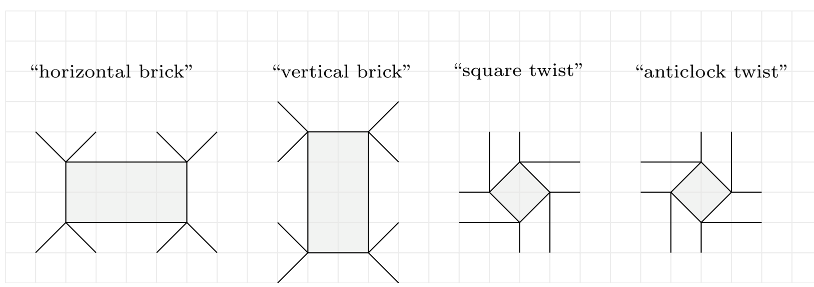


FIGURE 1. ways to think about basic units of brickwall/square twist/square weave crease pattern

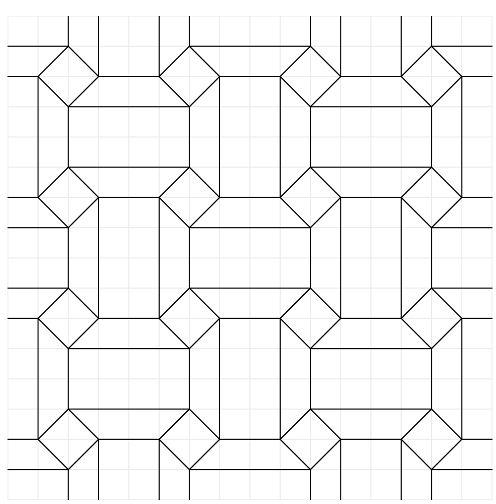


FIGURE 2. crease lines for brick wall [M], and other origrami tessellations, without directions shown

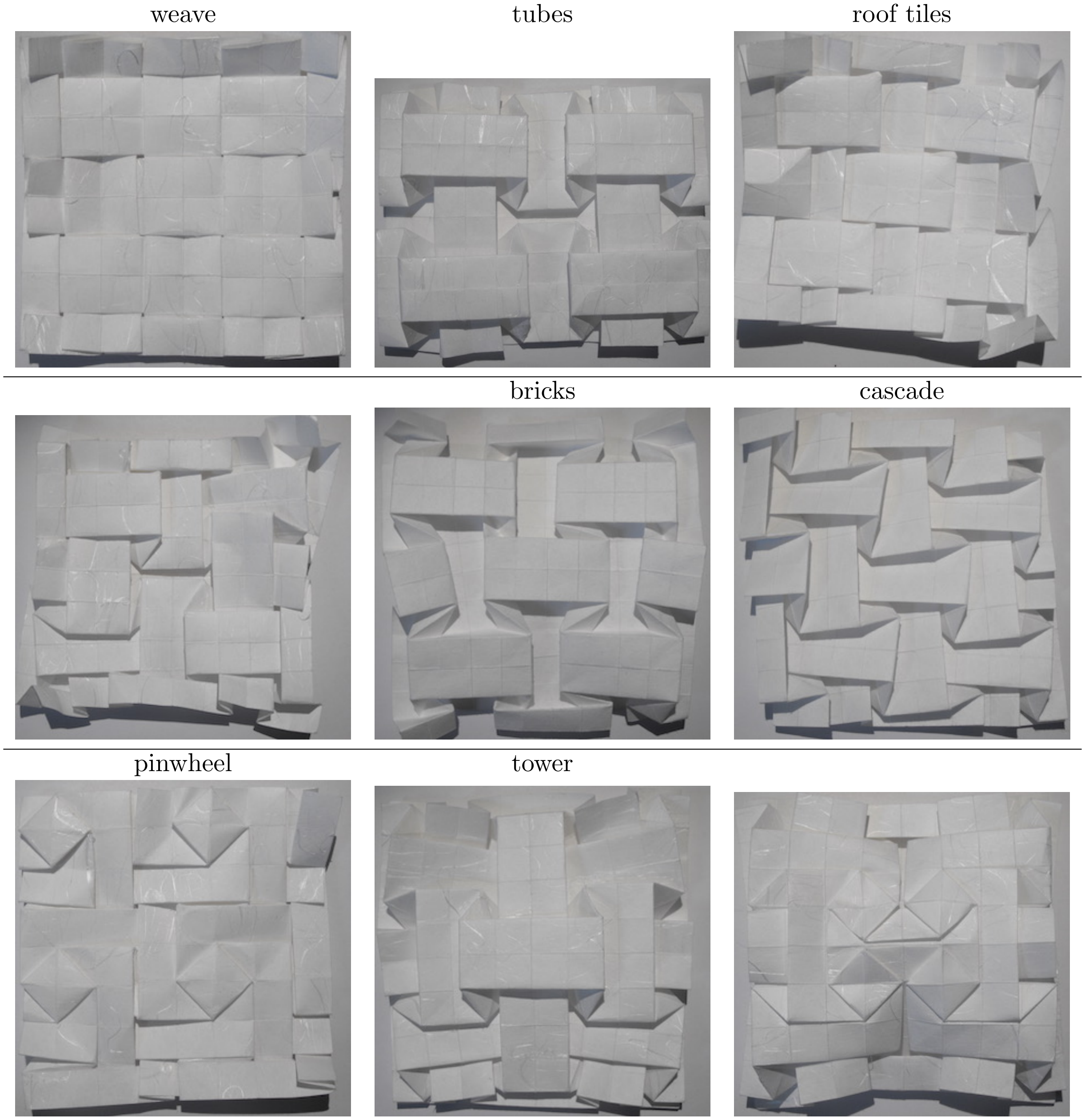


FIGURE 3. Some ways of folding crease pattern in Figure 2

1. COUNTING

I really like how some of these turned out, especially the "roof tile", "cascade", "tubes", "tower", "pinwheel". I probably missed some other nice looking variations.... to find out, I wanted to enumerate all possible ways of choosing crease directions for Figure 2, and fold them all (or sketch, then fold anything interesting looking). These patterns could also be useful as starting points for other tessellations, as was done in [V], so it's good to have a complete list.

Also, given any flat origami tessellations, it's likely that there will be many ways to change the directions of the folds and end up with quite different looking tessellations, doing this for the brick wall pattern just gives an idea of what might be possible.

1.1. **counting folds at a vertex.** In order to count how many crease patterns there are, first consider how many different ways there are to fold at any vertex. Each vertex is the same up to symmetry. Figure 5 shows the creases from one vertex. The two crease lines labeled a and b must have different directions, otherwise the sections of paper A and B would have to go through each other, which is not allowed. So, a and b are either mountain/valley or valley/mountain. And then this determines the direction of c and d, which have to be as shown in Figure 5. So there are 4 cases. I want to think of these cases as depending on the crease direction of creases b and c, since

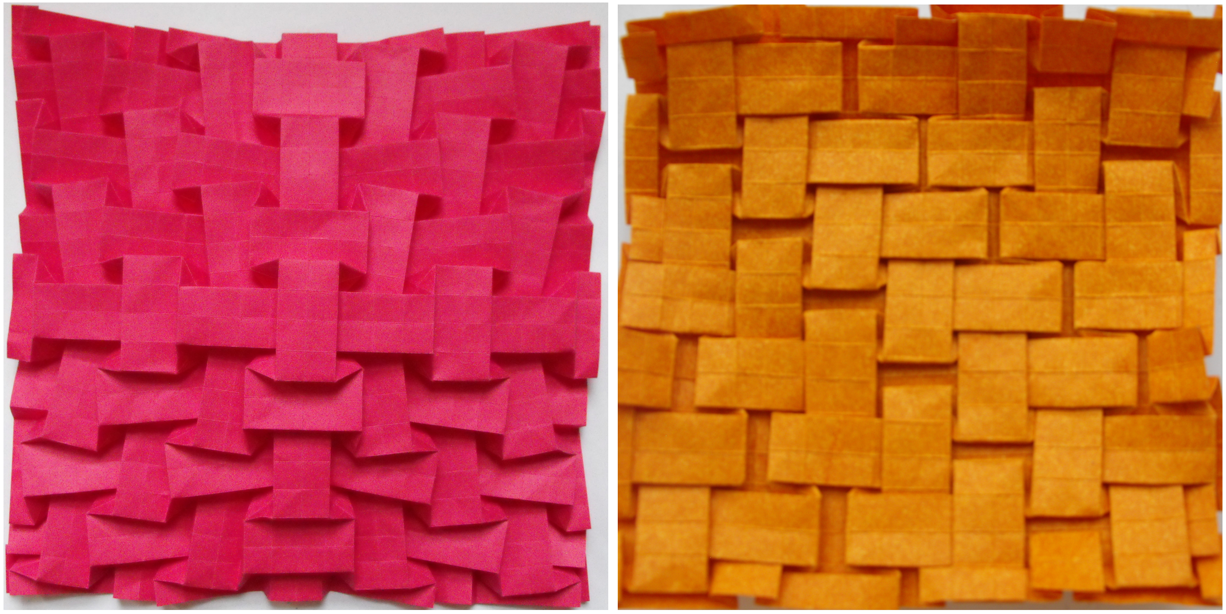


FIGURE 4. “tower” of piled up bricks, and an approximation to a herringbone pattern

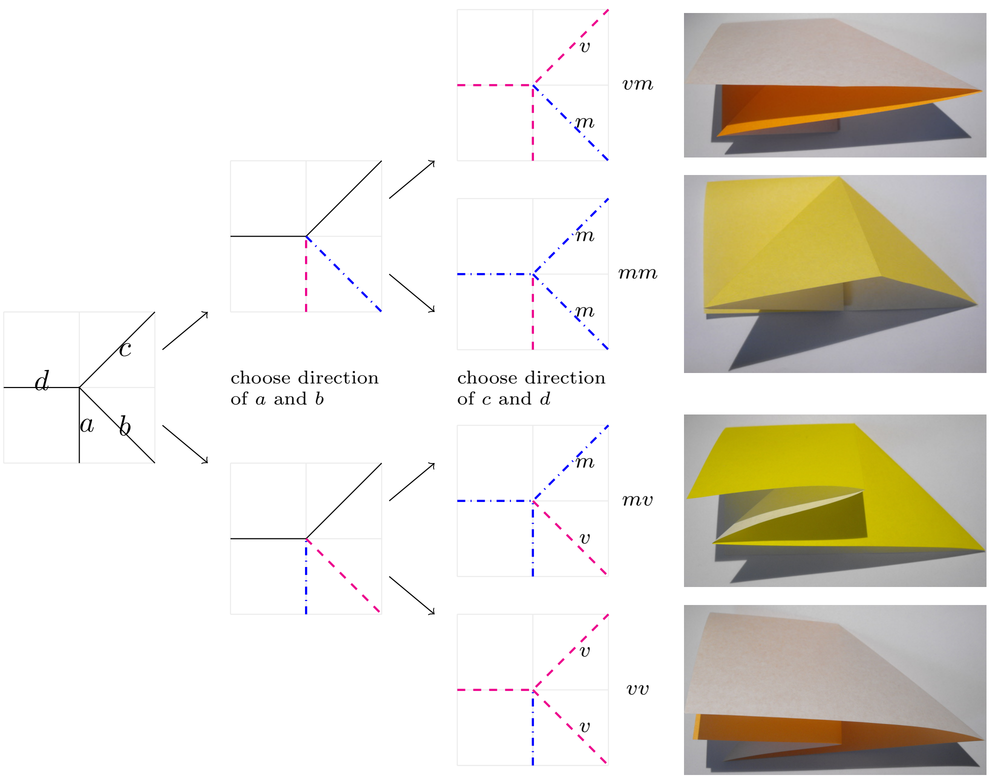


FIGURE 5. crease direction at a vertex; these are determined by crease directions of a and b, but labeled by crease directions of c and b — “m” for mountain and “v” for valley

these are sides of the twisted squares in the tessellation, and I want to next deterimine the possible ways of folding these. I’m choosing to make creases depend on the squares, rather than the rectangles, since there are 16 squares in Figure 2, all completely on the paper; there are parts or all of 25 rectangles, so better to work with squares. However, when I was doing the folding, I was thinking about where the rectangles would end up, and how they would be either under or over each other.

1.2. **counting folds of a square.** Since the crease directions at a vertex are determined by the crease directions of the sides of a square, we just have to choose crease directions of the square, which determines the rest. Figure 6

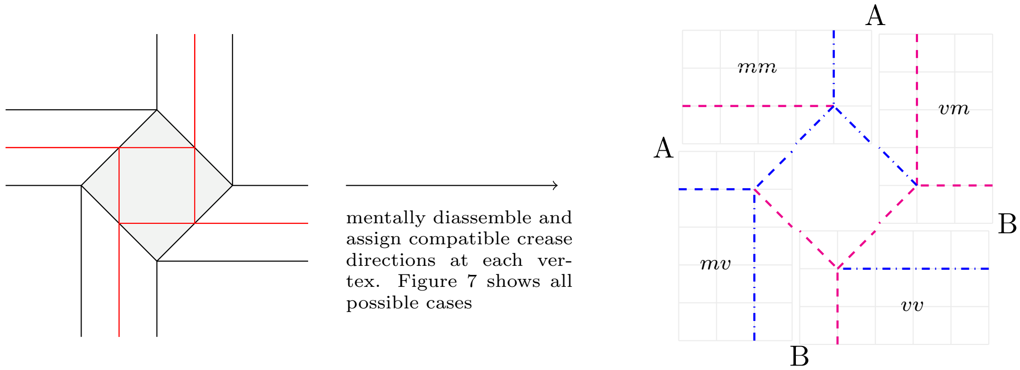


FIGURE 6. Example of crease direction at a vertex (taken from Figure 5) to crease direction of square; red lines cut apart square to vertex pieces, leaving a gap in the middle. Folded view on right, labeled along edges with (A) mountain and valley or (B) valley and mountain. So this square unit is labled ABBA.

shows an example of how the crease directions at vertices determine the crease directions of the square unit. Up to symmetries there are 6 ways of assigning crease directions to the square twist unit, which are shown in Figure 7.

For any one of these ways, the pattern can be repeated over and over, simply by reflecting in the edges of the unit, as for example, shown in Figure 9, where repeats of the fourth fold, labeled "cascade" in Figure 7 give the "cascade" pattern shown in Figure 3.

Note that the creases of the square unit are determined by the crease directions of the folds meeting the edges of the unit; there are only 2 possible cases - these folds (short and long lines respectively) can be (A) mountain and valley or (B) valley and mountain. For example, as labeled in Figure 6 around the side of the unit. We can lable each square with a corresponding sequence of As and Bs, starting from the top, and labeling clockwise.

In Figure 7 we showed the 6 ways of folding the single square unit, up to rotation and reflection.

Now we consider how many ways to fold creases on a four unit configuration, as in Figure 10. Each of D_i for $i = 1, \ldots, 12$ has to be assigned to be either A or B. So there are a total of $2^{12} = 4096$ ways to do this. However, this is not taking into account symmetries. A pattern with no symmetries will have 3 other patterns which are the same up to symmetry (rotation and reflection), so if no patterns had symmetries, we would have $4096/4 = 1024$ cases. Actually there will be more thsn this, which we can compute using Burnsides Lemma. However, since this is quite a lot, I'm going to make further assumptions about the crease directions, namely that the top and bottom creases are the same, i.e., as in on the right in Figure 10. So now we expect roughly $2^8/4 = 64$ cases. We can also take the symmetry of turning the piece over or switching crease directions, so including this gives us now approximately 32 cases. These cases can be considered as rotations through 180° about axes lying in the plane of the paper, O_1, O_2, shown in Figure 11, or S, switching all crease directions, i.e., $A \leftrightarrow B$, or T switching crease directions, and rotating through 180°.

Our symmetry group of this crease diagram has 8 elements $I, H, V, R, S, T, O_1, O_2$.

To apply Burnside's Lemma, we need to look at the crease patterns invariant under the different symmetries

Burnside's Lemma says that

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} X^g$$

So, plugging in the numbers in Figure 11 we find that in this case

$$|X/G| = \frac{1}{8}\left(2^8 + 2^6 + 2^6 + 2^4 + 2^4\right) = 52$$

Or, in case we don't allow turning over, we get

$$|X/G| = \frac{1}{4}\left(2^8 + 2^6 + 2^6 + 2^4\right) = 100$$

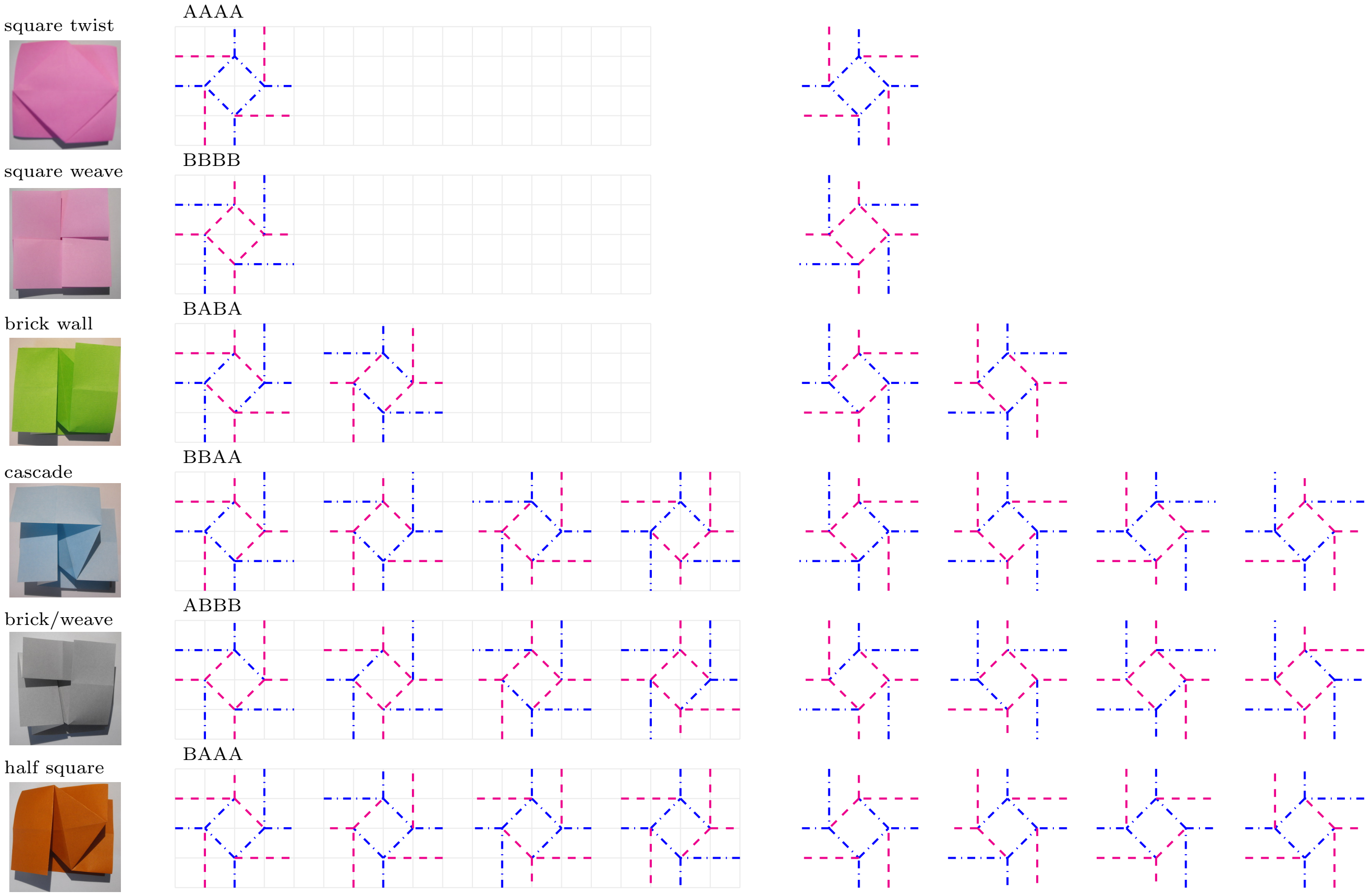


FIGURE 7. Crease directions of square; crease patterns the same up to rotation are in the same row; reflected versions are on the right. Photo matches something in row

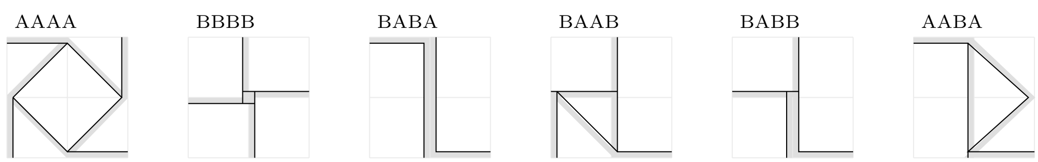


FIGURE 8. Folded units

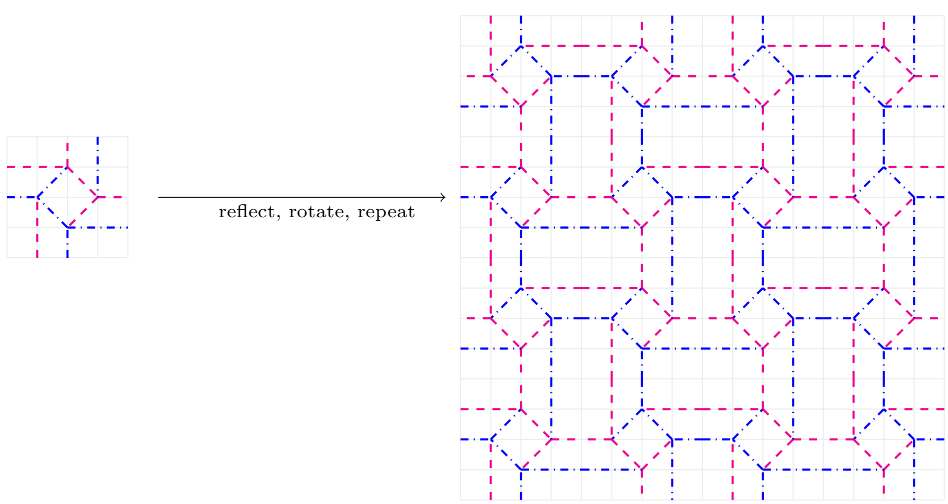


FIGURE 9. Translations and reflections of "cascade" crease unit from fourth row of Figure 7

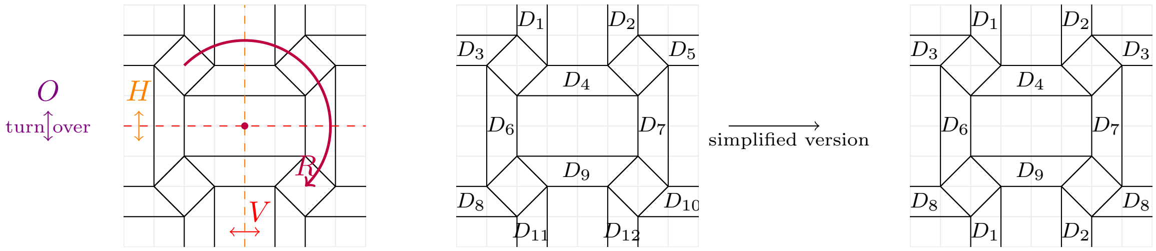


FIGURE 10. Determining crease directions for four units, depending on directions of sets of parallel pairs of folds. For simplicity, I'm going to confine consideration to the case on the right, which can be repeated by translations.

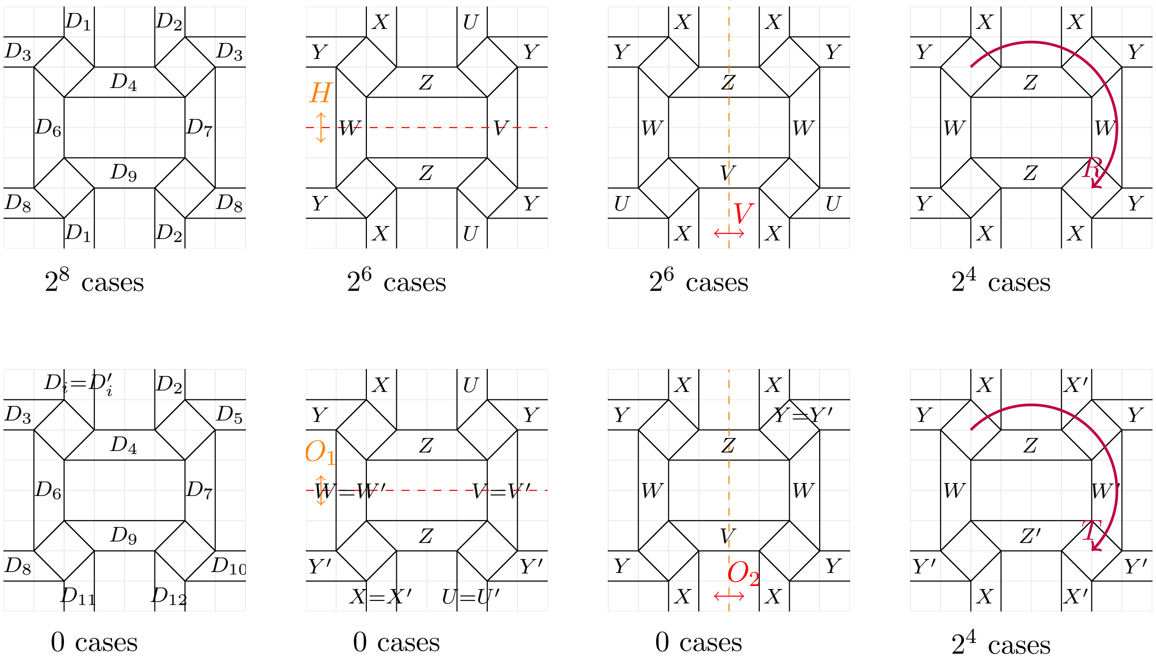


FIGURE 11. Crease patterns invariant under symmetries (simplified version, as in right in Figure 10). Here $A' = B$ and $B' = A$, so $X' = X$ is impossible

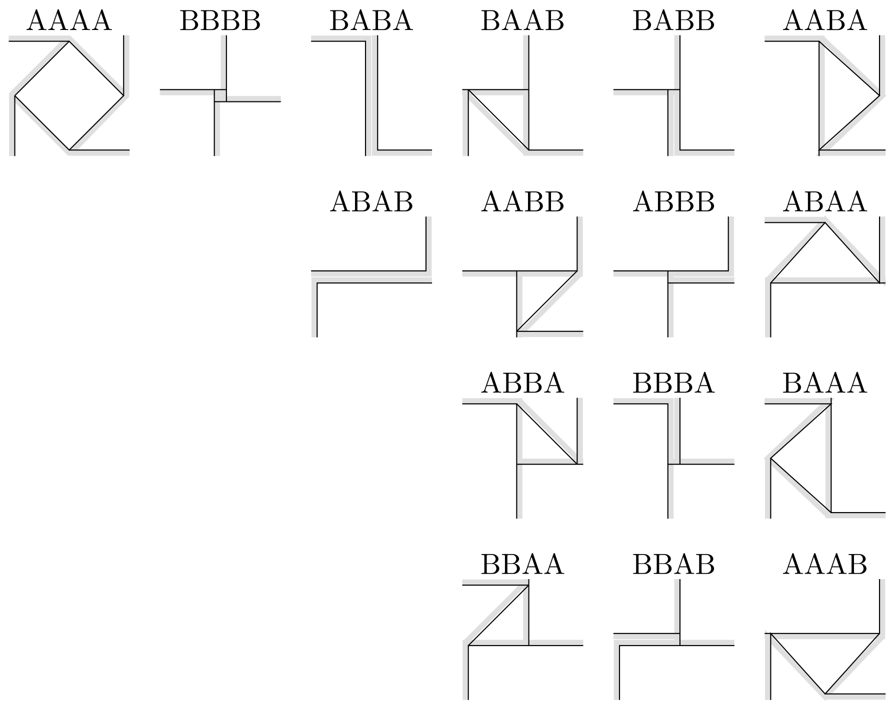


Figure 12. possible clockwise units

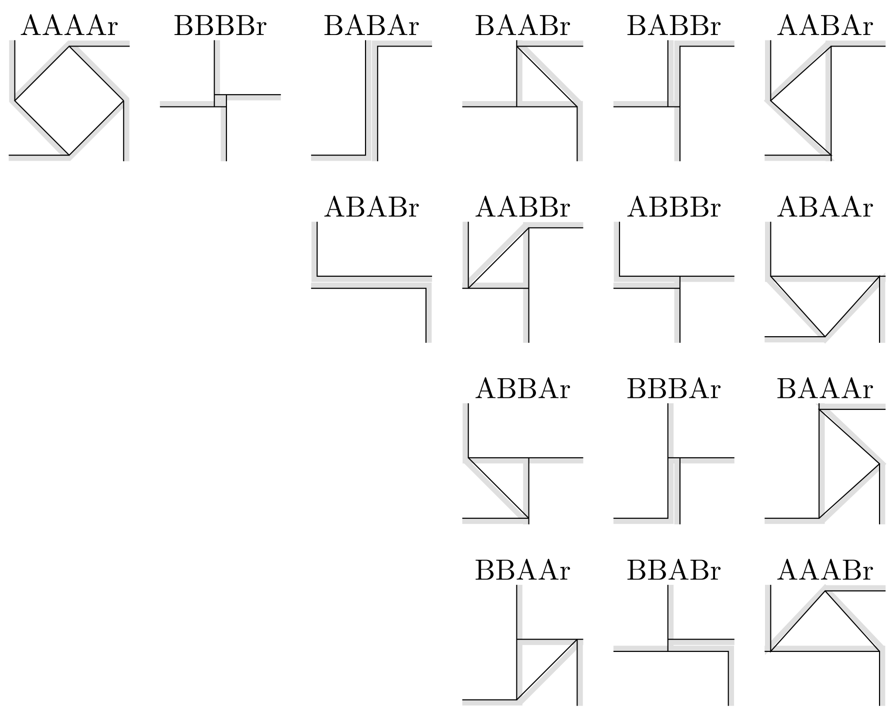


Figure 13. possible anti clockwise units

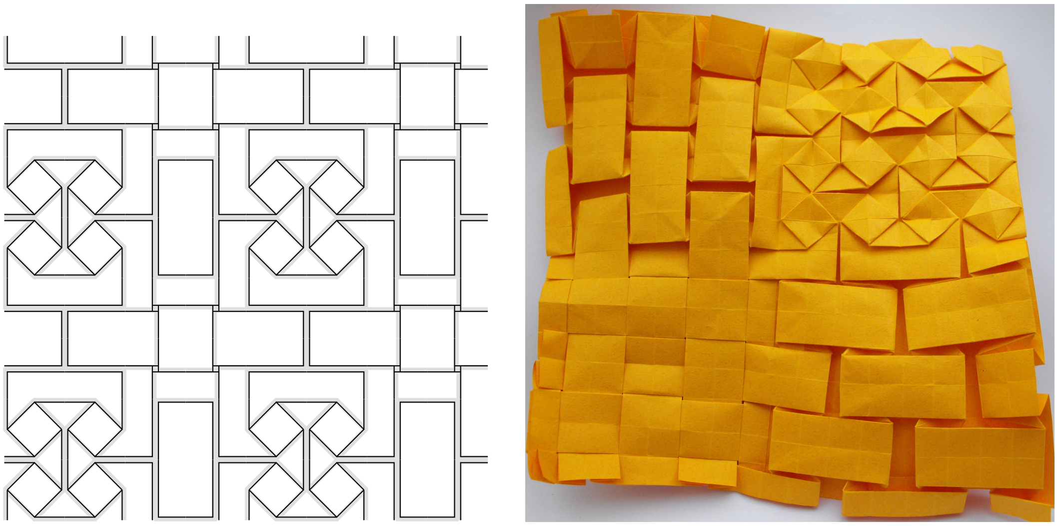
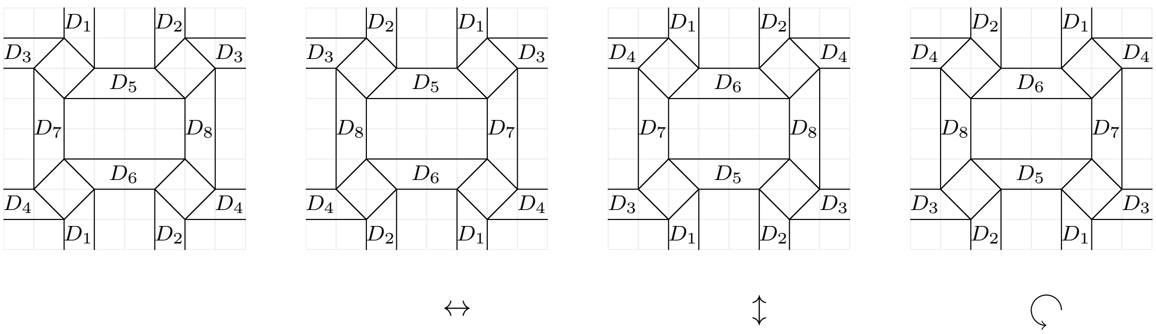


FIGURE 14. Example: transitioning between square twist, square weave, and brick wall



	↔	↕	↻
12345678	21345687	12346578	21436587

FIGURE 15. Labeling and symmetries

Now let's find all 100 patterns and draw diagrams of them. To do this with no repeats, lets put a total ordering on the set of lables, and then from the set of patterns which are equal up to symmetry, let's take a minimal representative pattern. We use an ordering with $A < B$.

The patterns found are all displayed below. In fact, some of them turn out to be the same, since even though they are not the same when just the smaller unit is give, when they are repeated, they may be the same after a translation. I have not taken this into account, so there are actually repeats amongst the 100 patterns shown below.

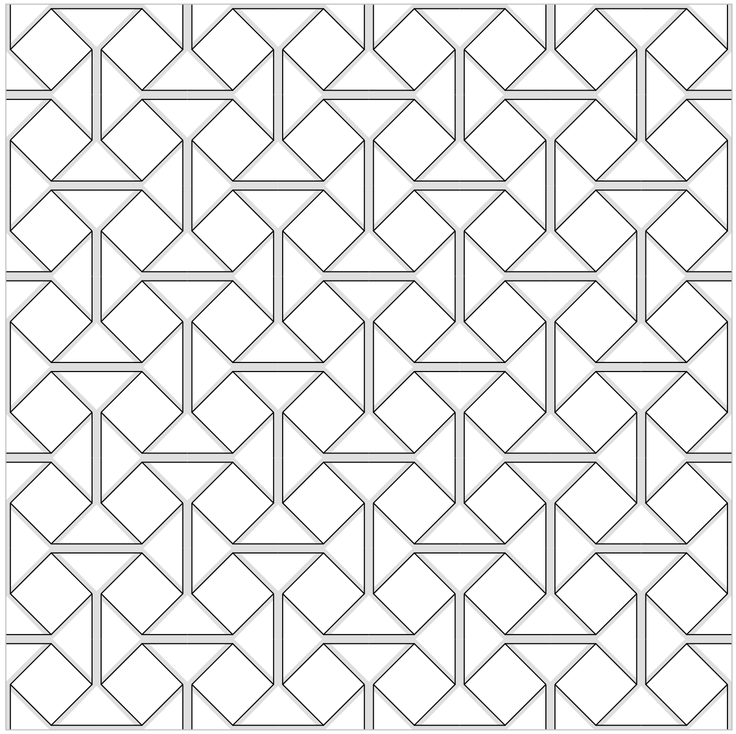


Figure 16. AAAAAAAA

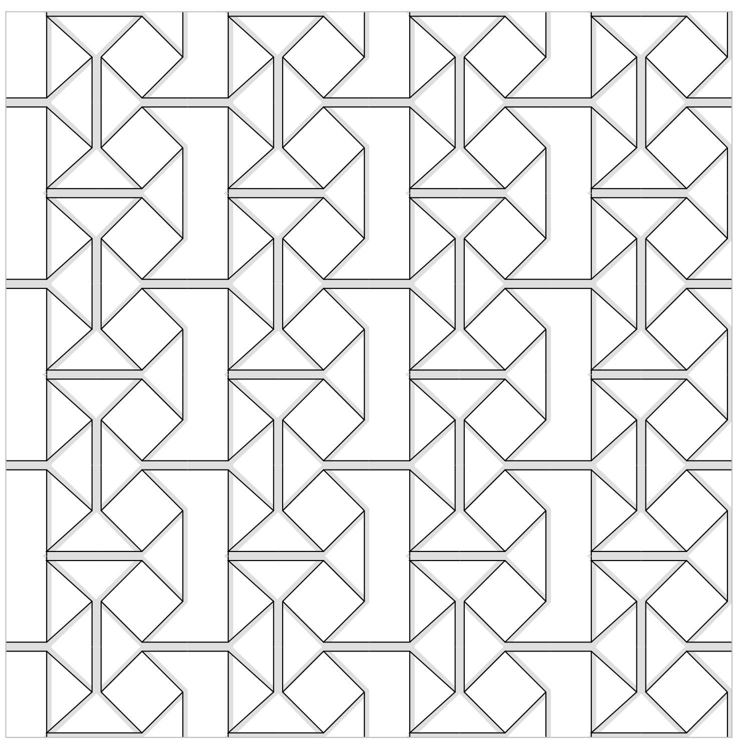


Figure 17. BAAAAAAA

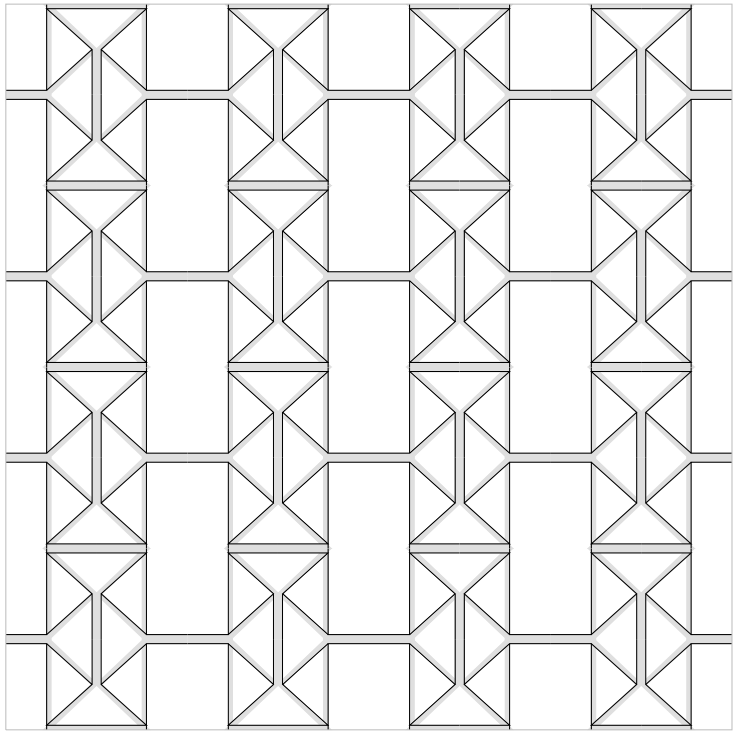


Figure 18. BBAAAAAA

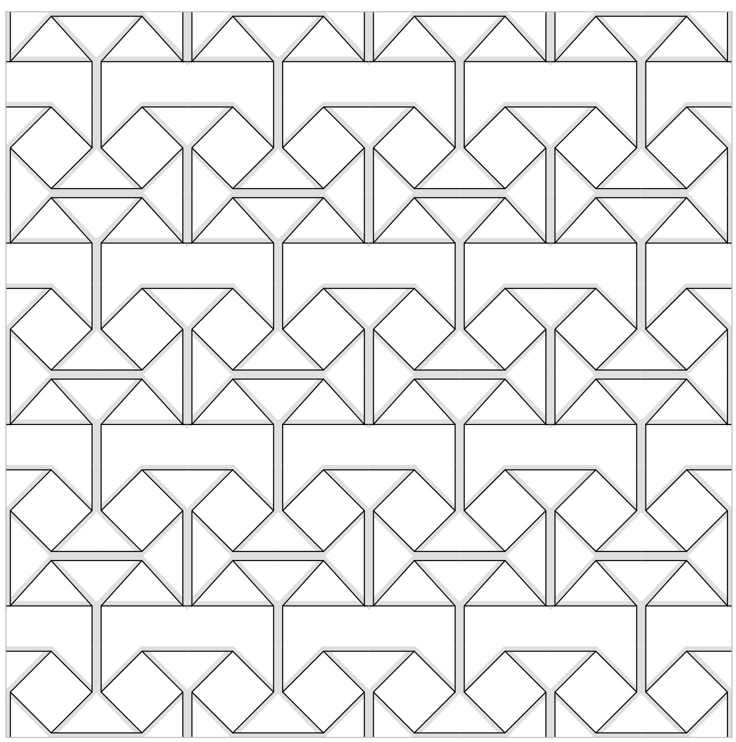


Figure 19. AABAAAAA

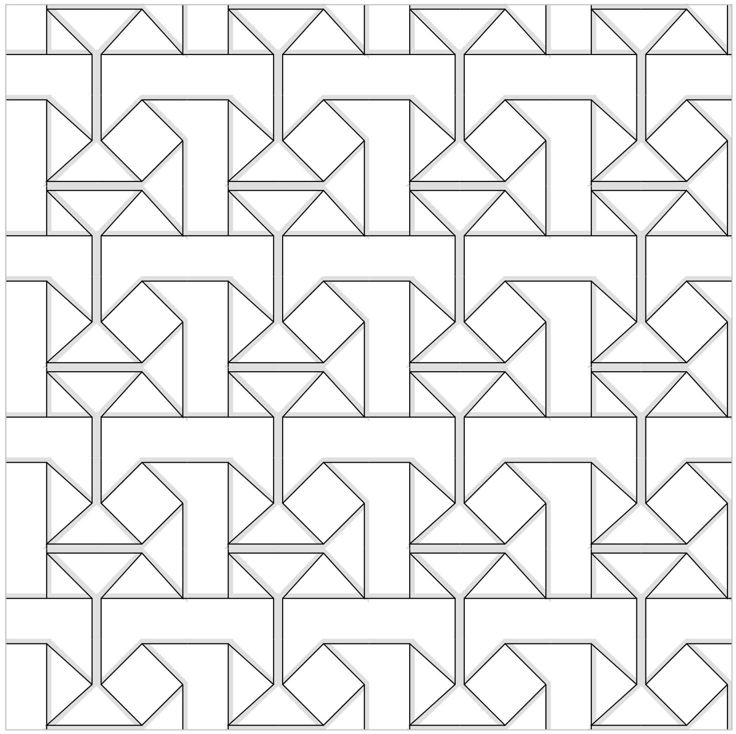


Figure 20. BABAAAAA

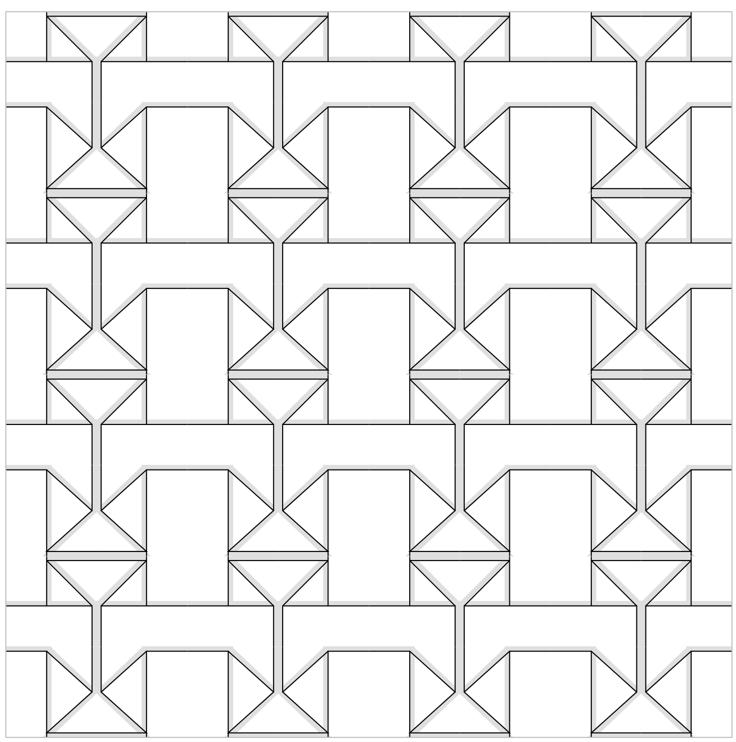


Figure 21. BBBAAAAA

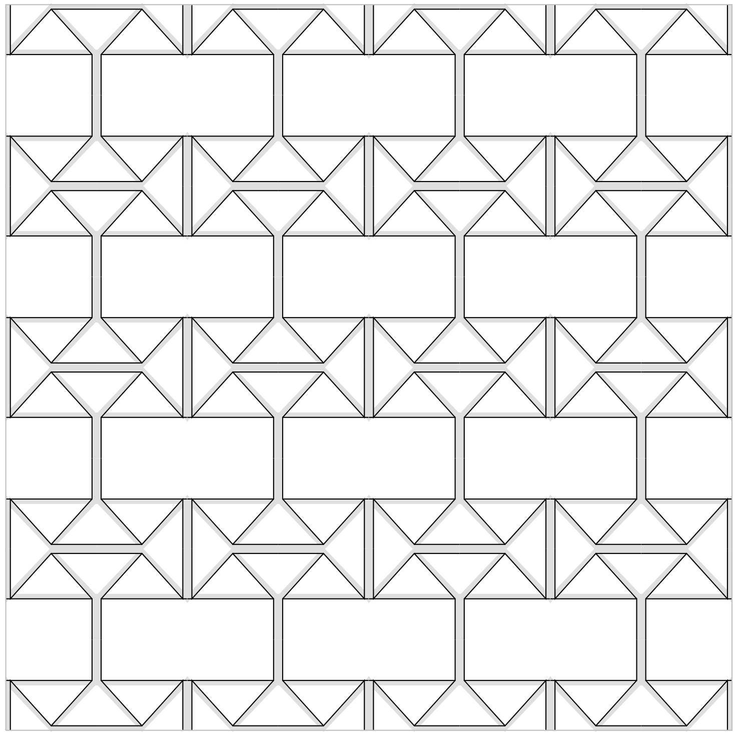


Figure 22. AABBAAAA

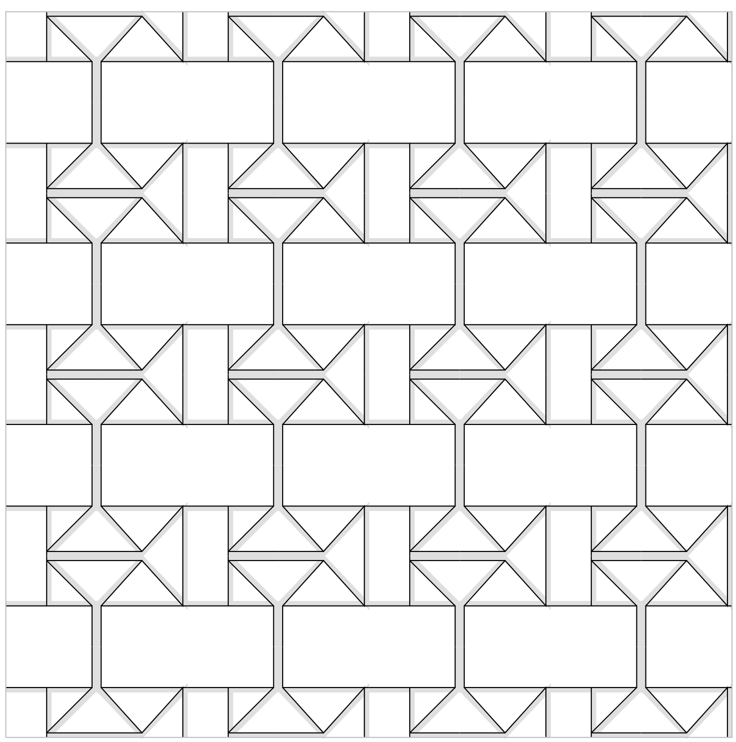


Figure 23. BABBAAAA

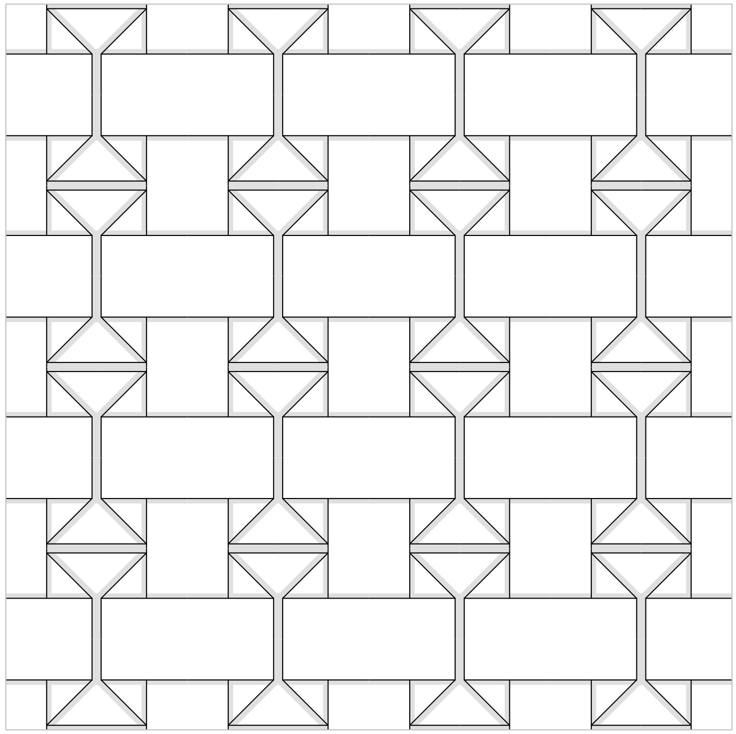


Figure 24. BBBBAAAA

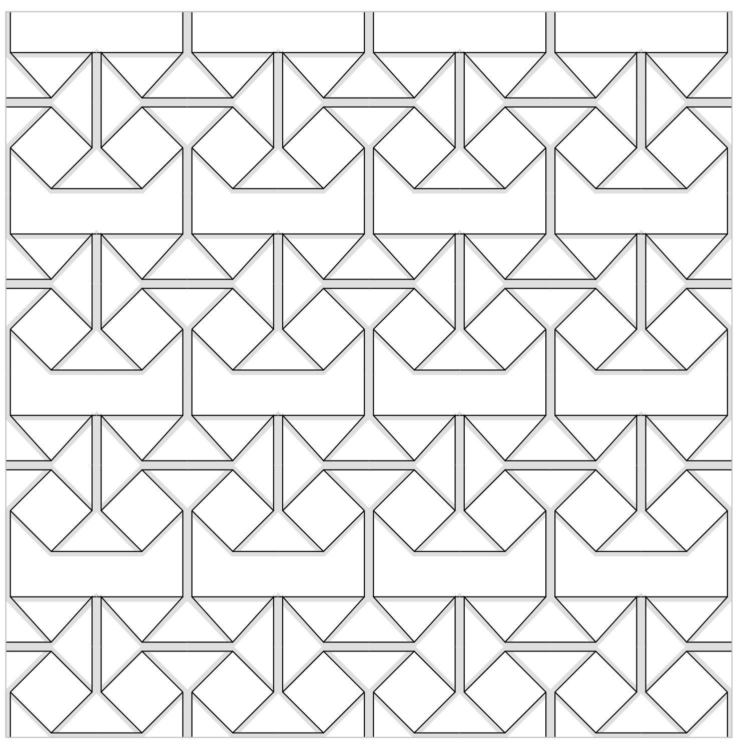


Figure 25. AAAABAAA

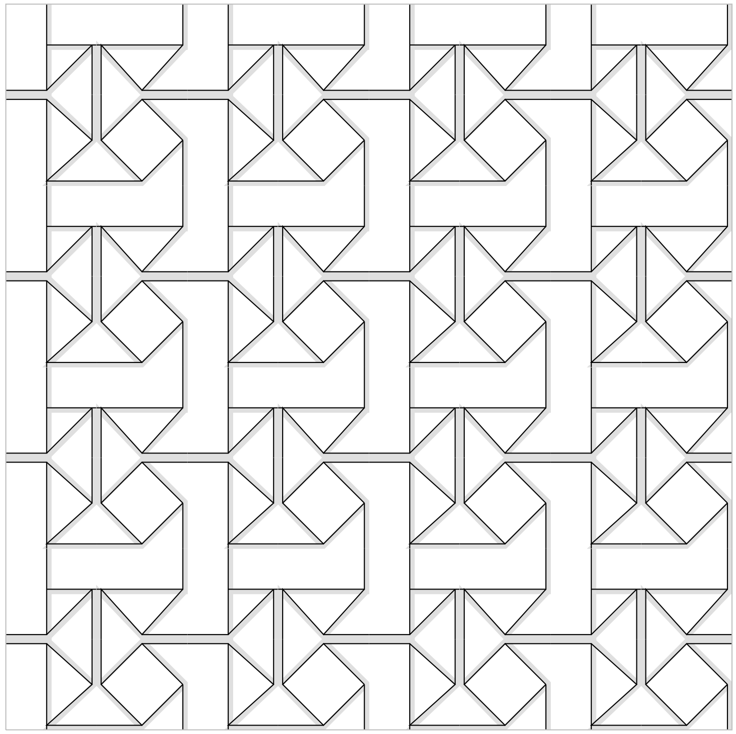


Figure 26. BAAABAAA

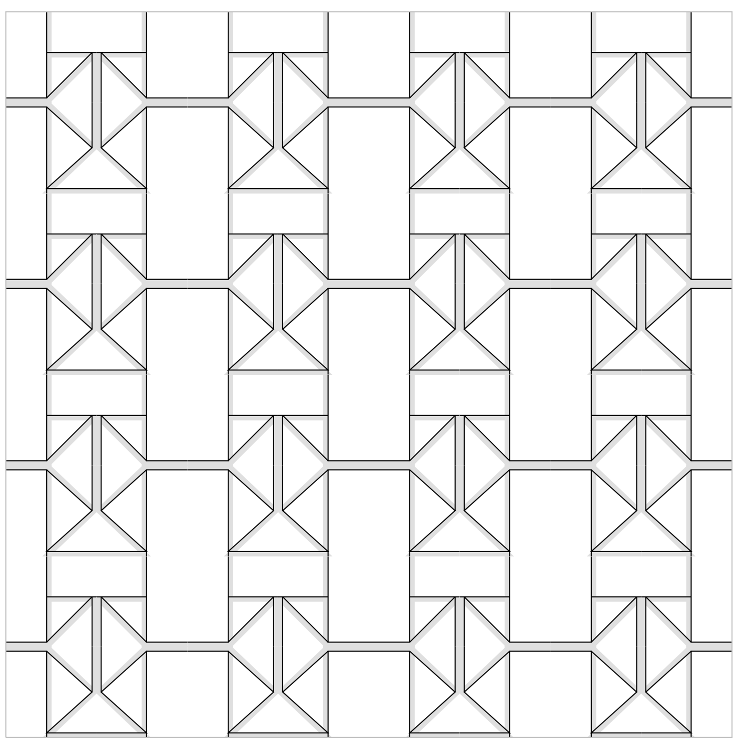
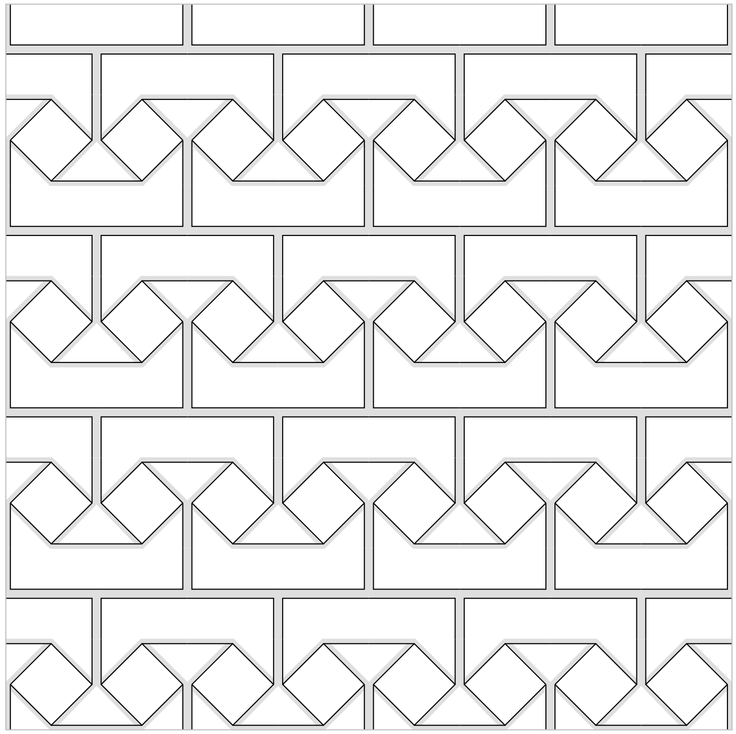


Figure 27. BBAABAAA



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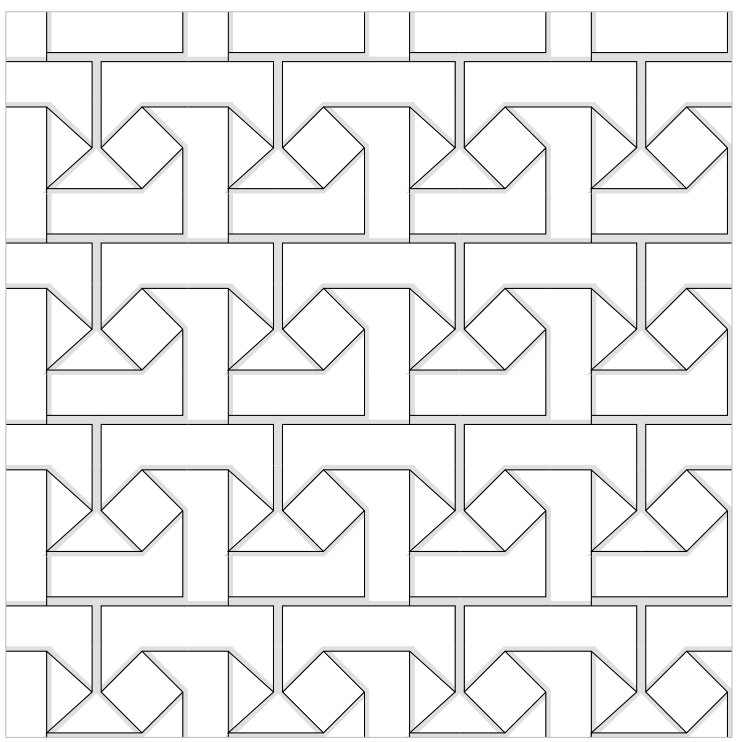
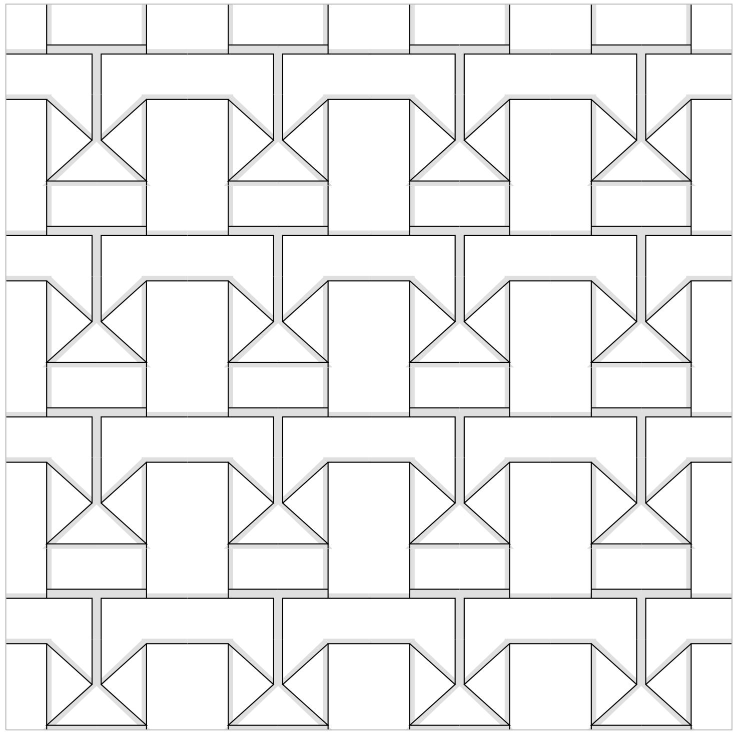


Figure 29. BABABAAA



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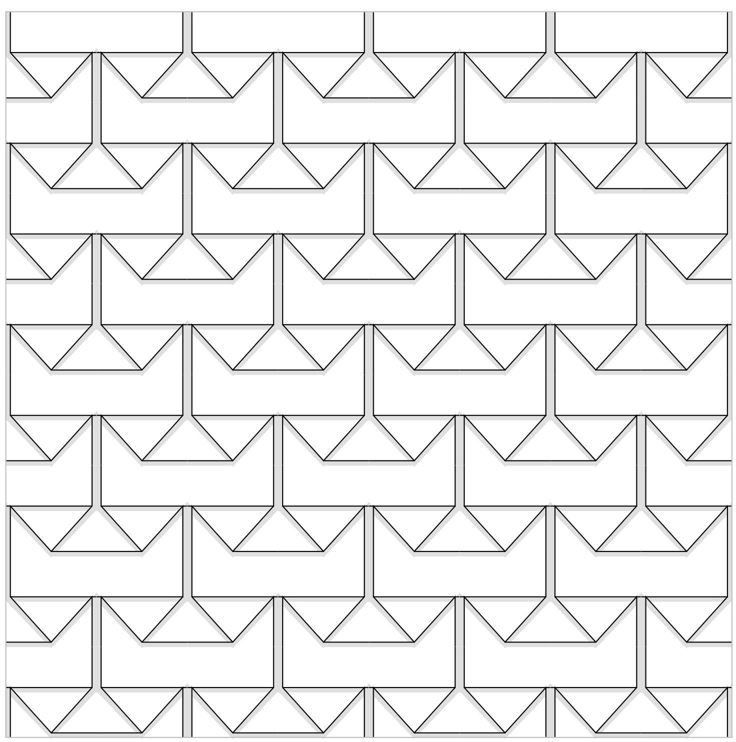

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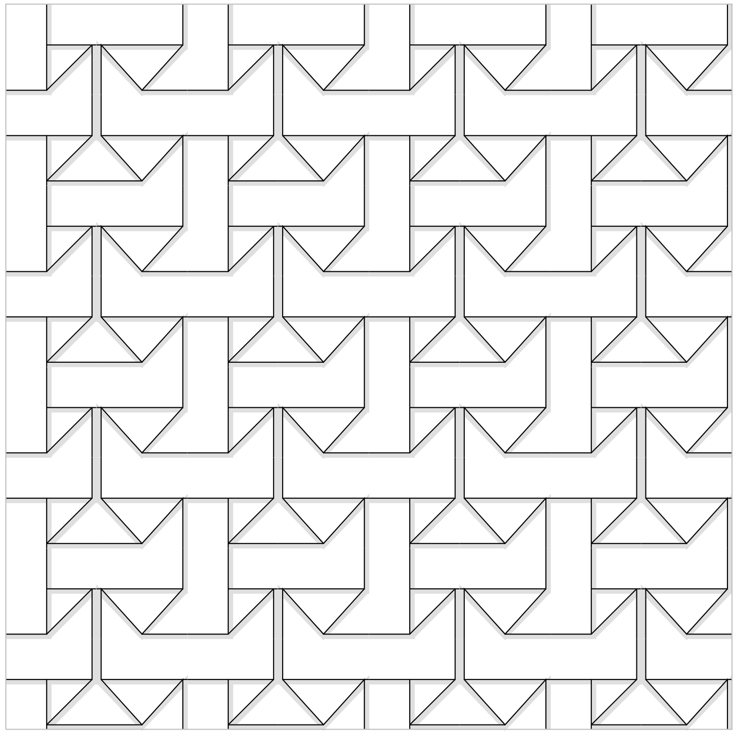

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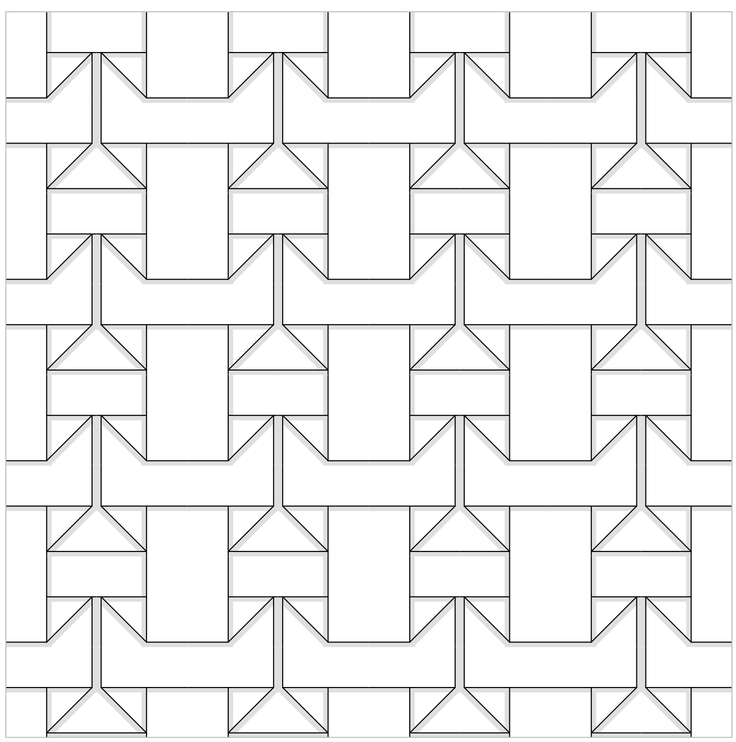


Figure 33. BBABBAAA

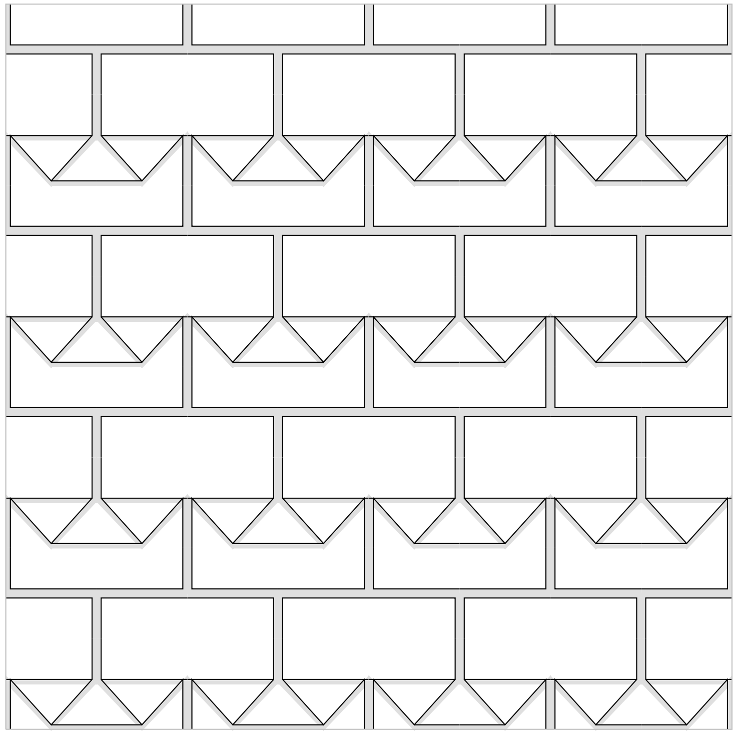


Figure 34. AABBBAAA

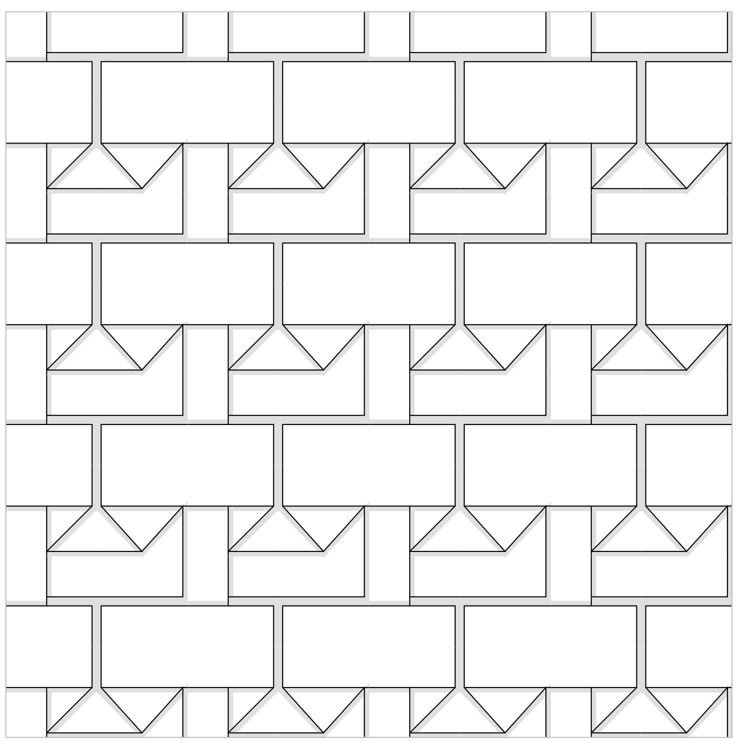


Figure 35. BABBBAAA

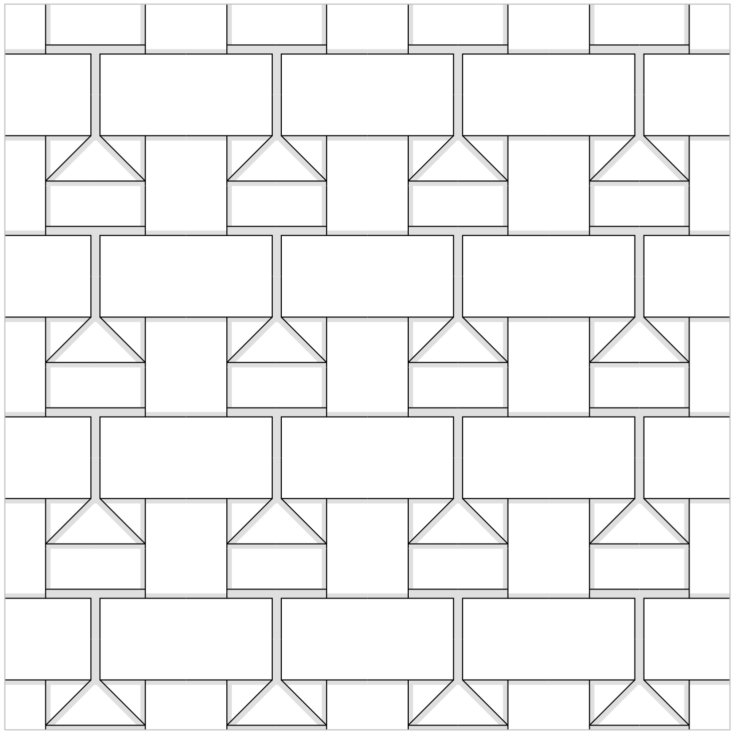
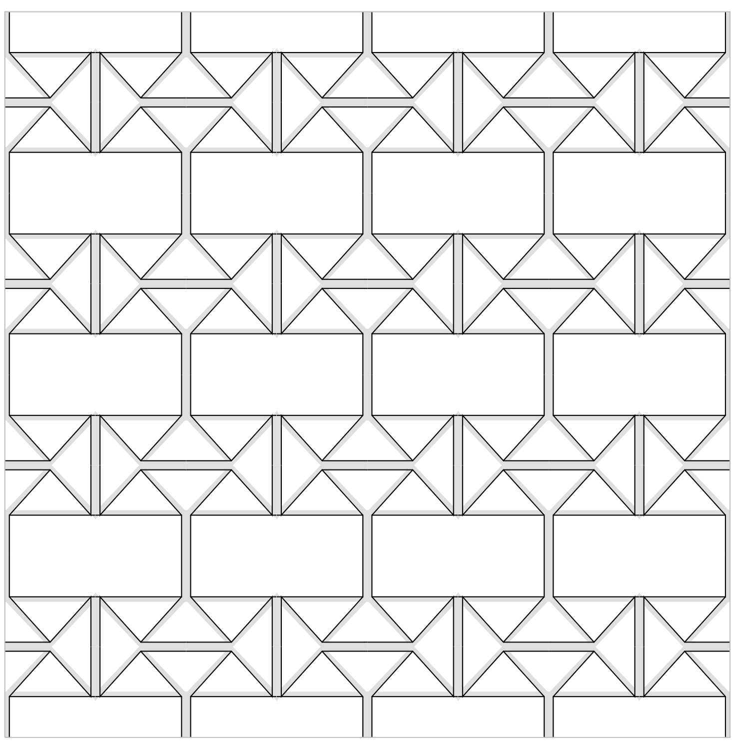


Figure 36. BBBBBAAA



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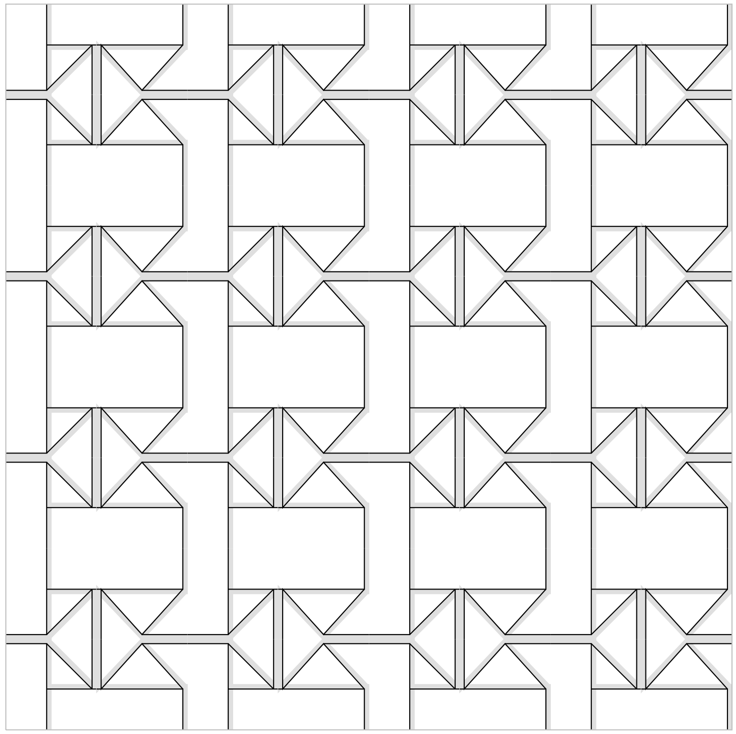


Figure 38. BAAABBAA

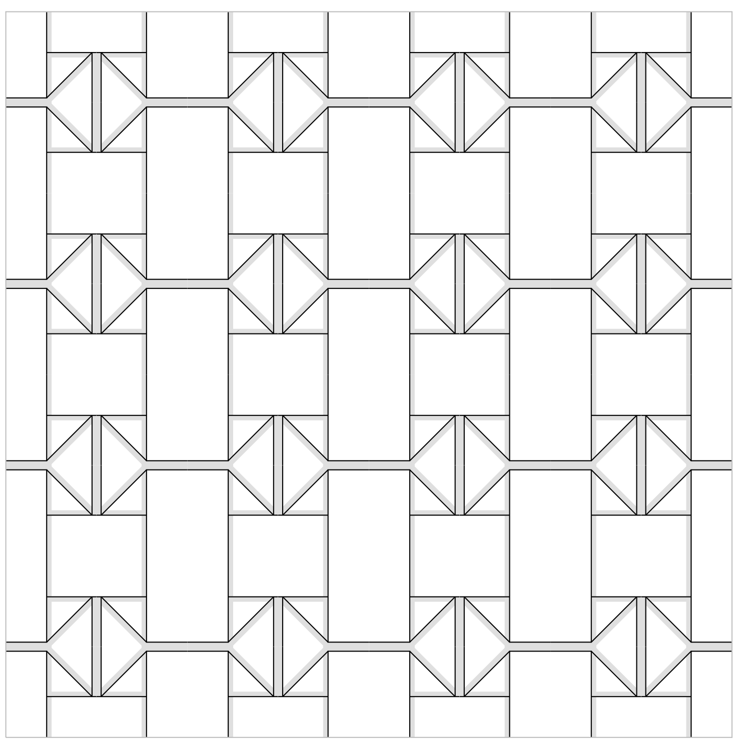


Figure 39. BBAABBAA

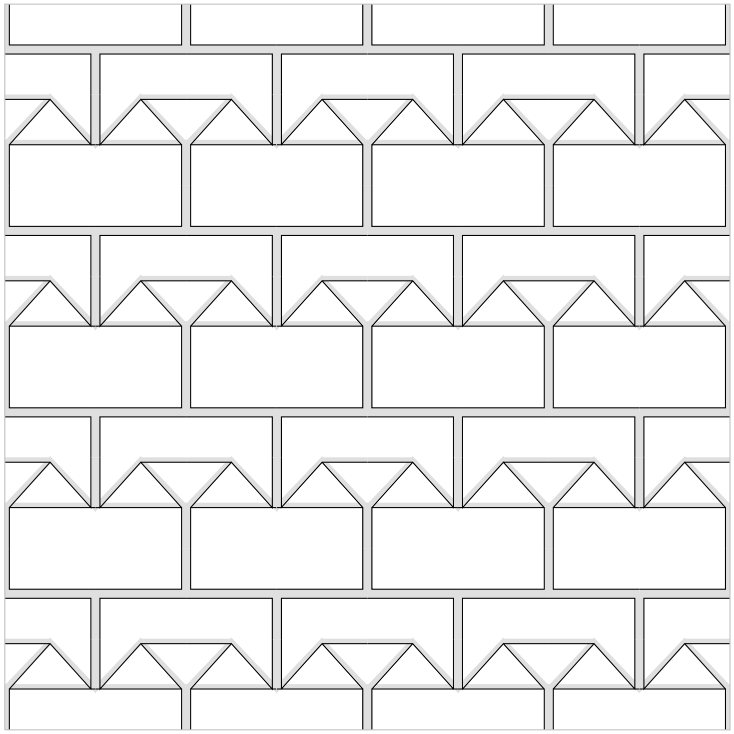
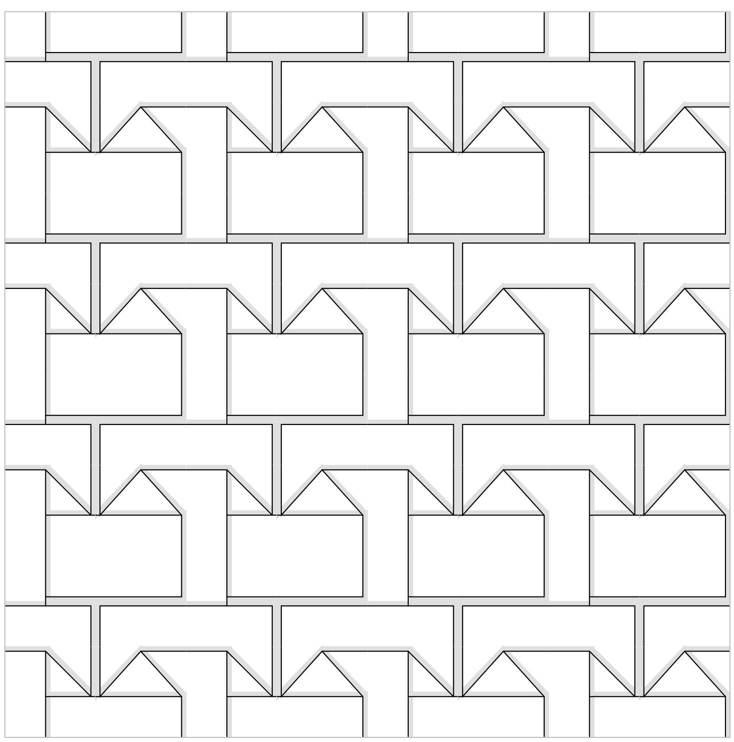


Figure 40. AABABBAA



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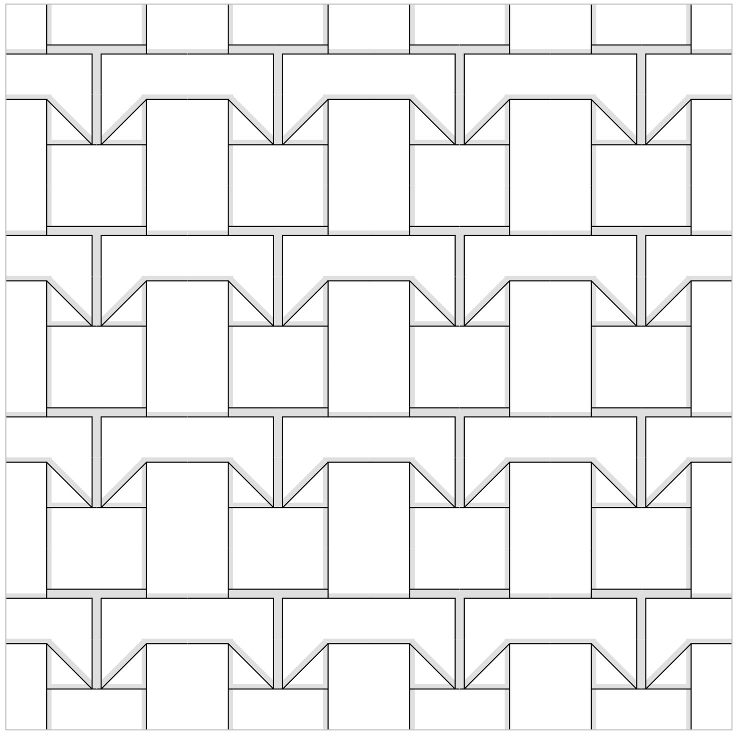

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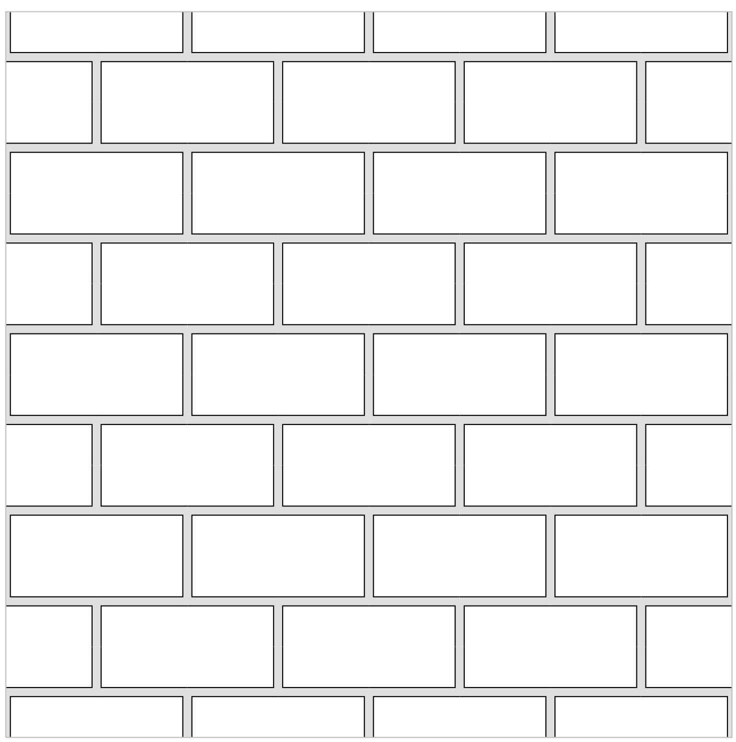

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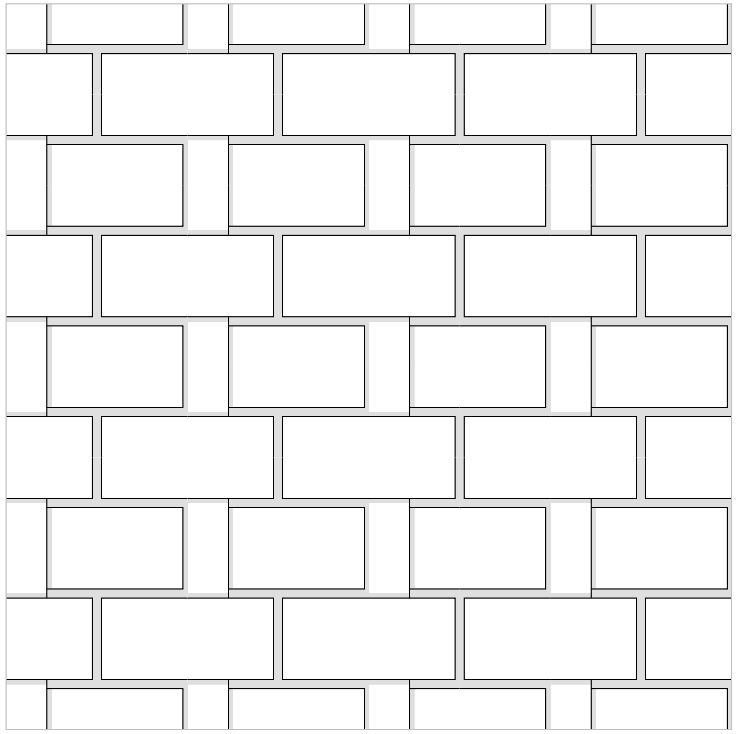

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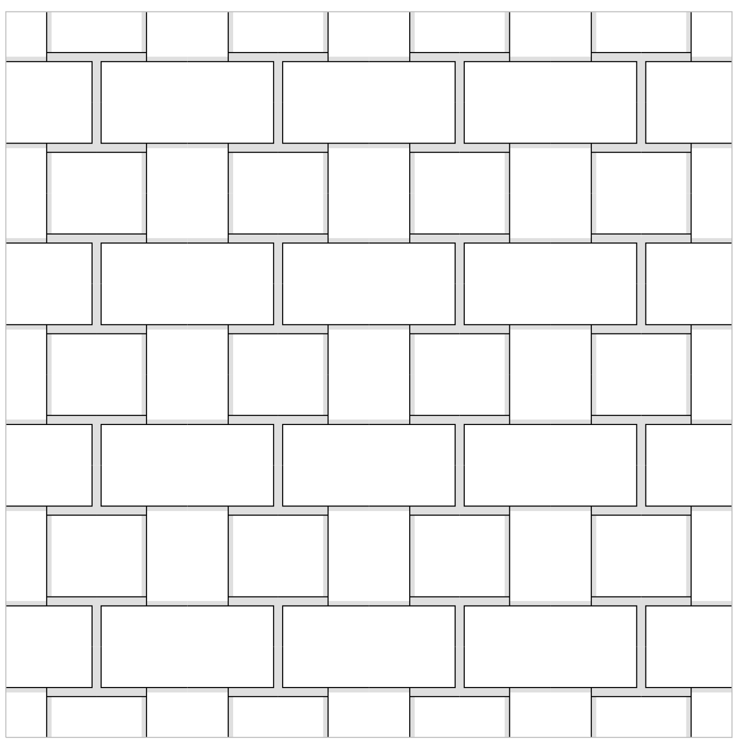


Figure 45. BBBBBBAA

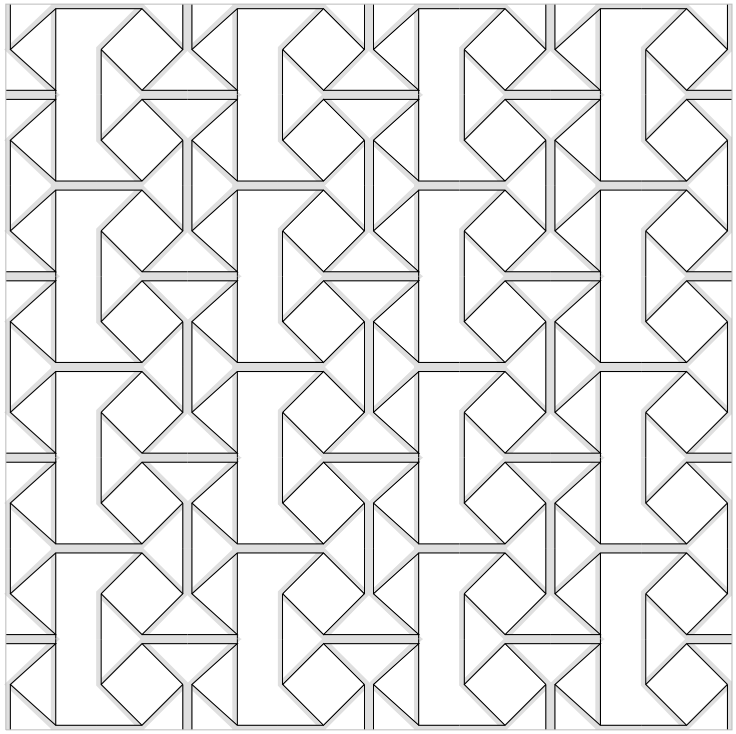


Figure 46. AAAAAABA

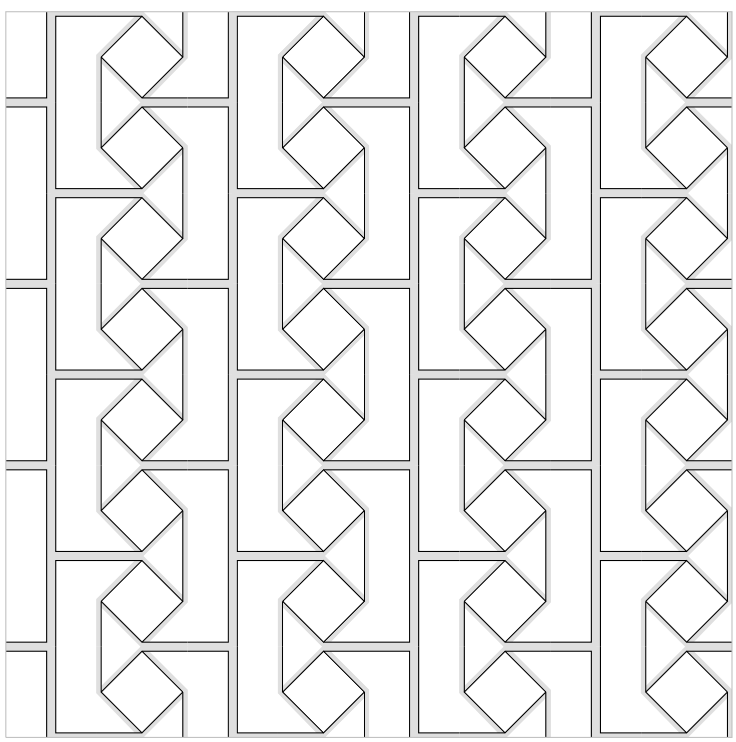


Figure 47. BAAAAABA

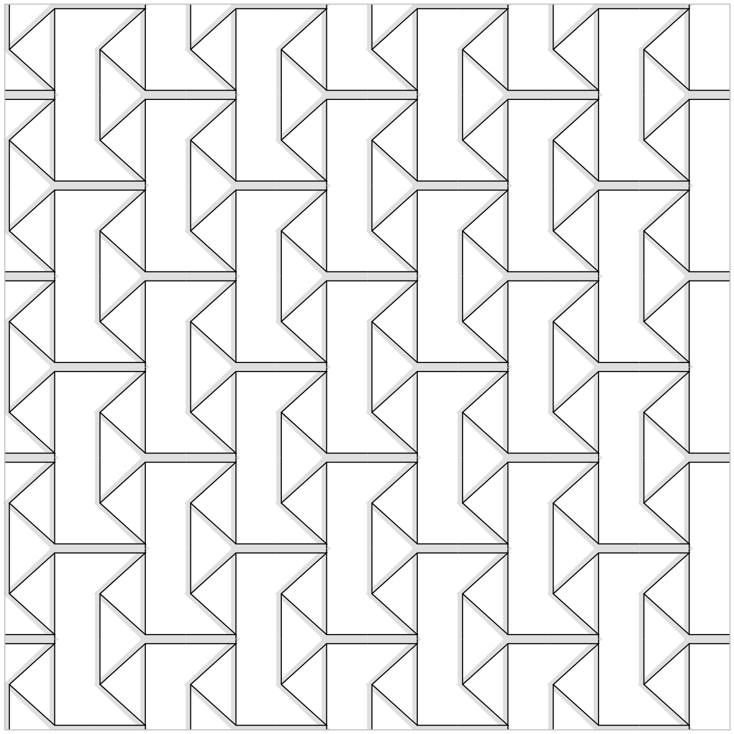


Figure 48. ABAAAABA

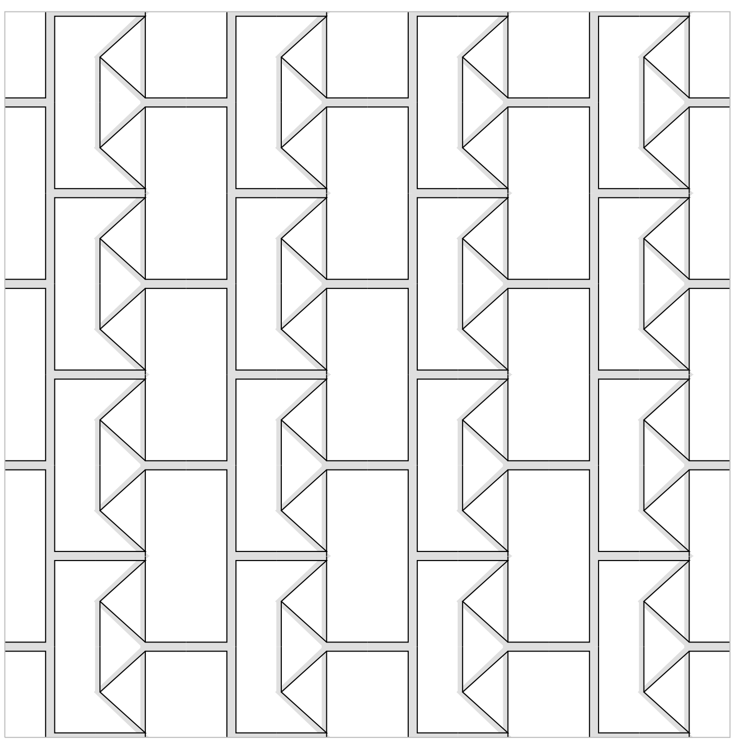


Figure 49. BBAAAABA

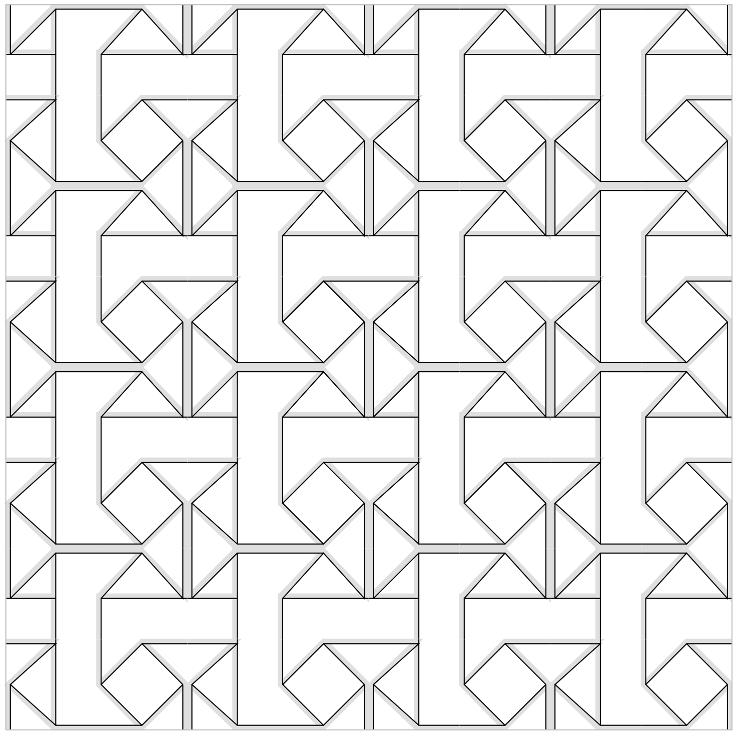
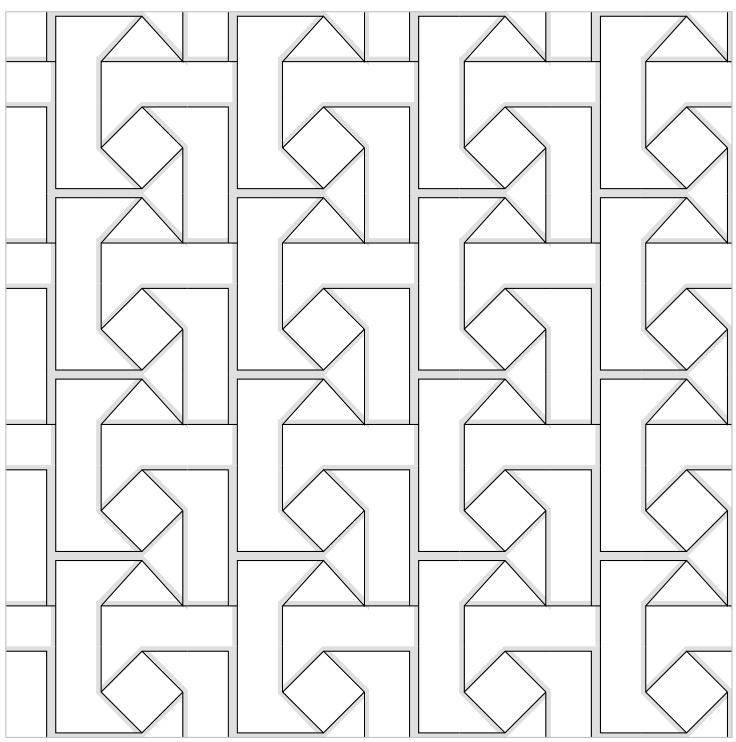


Figure 50. AABAAABA



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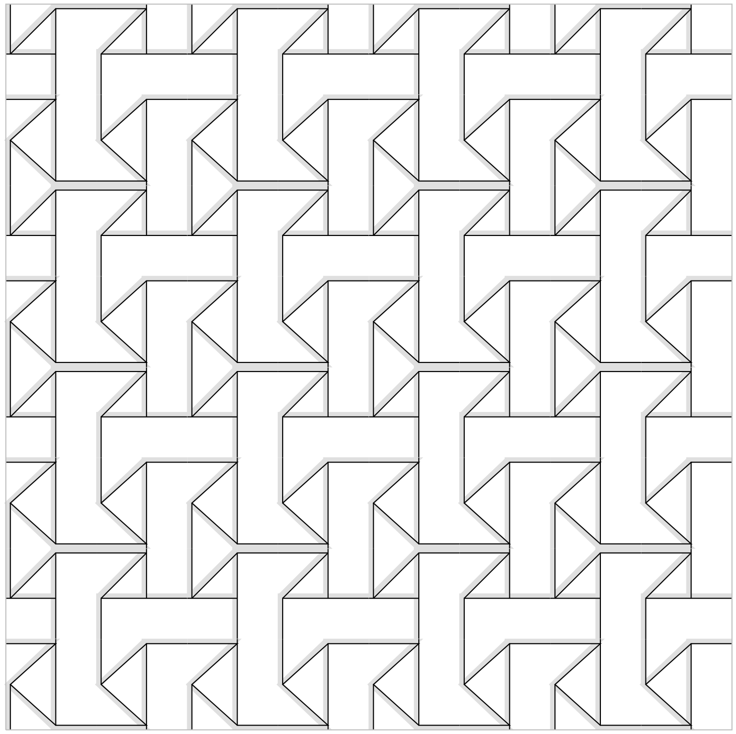


Figure 52. ABBAAABA

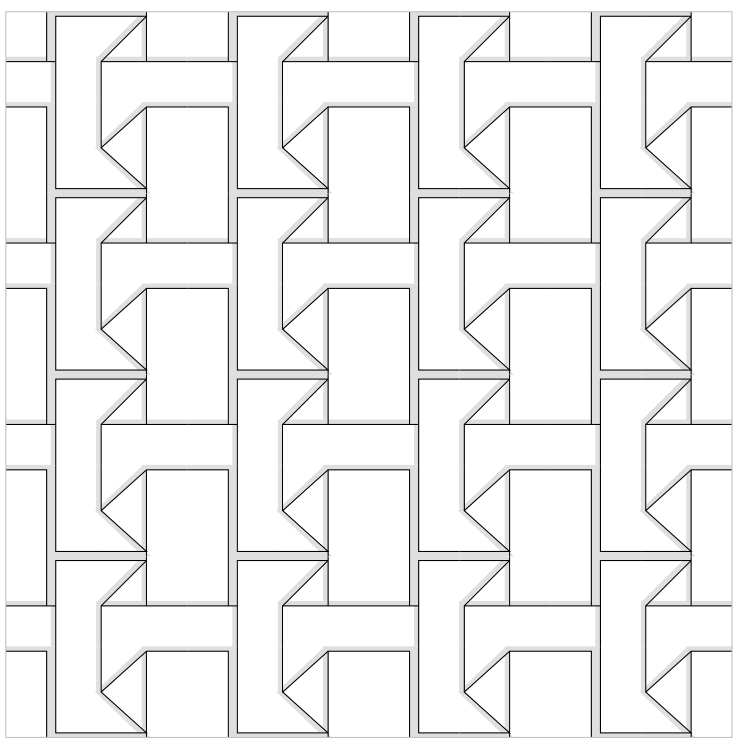
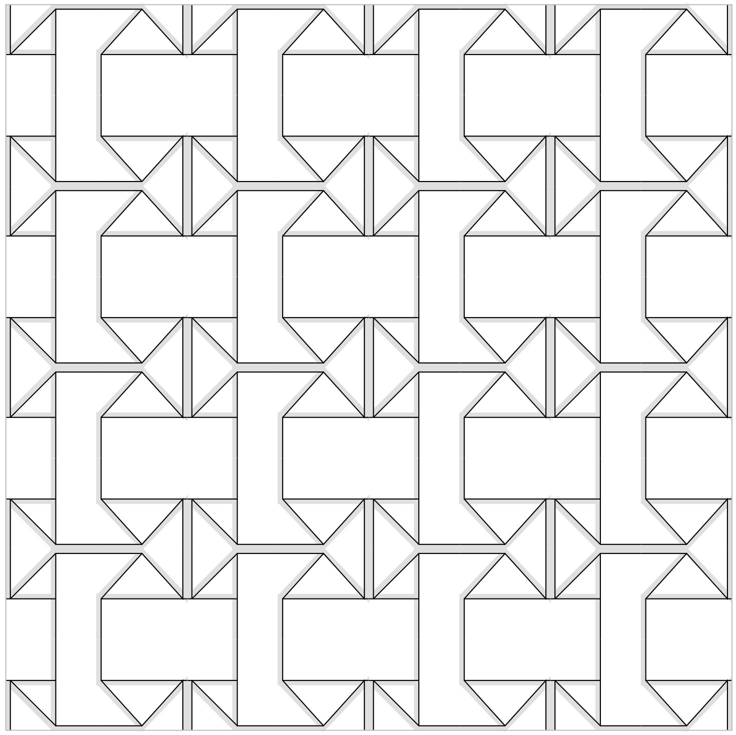


Figure 53. BBBAAABA



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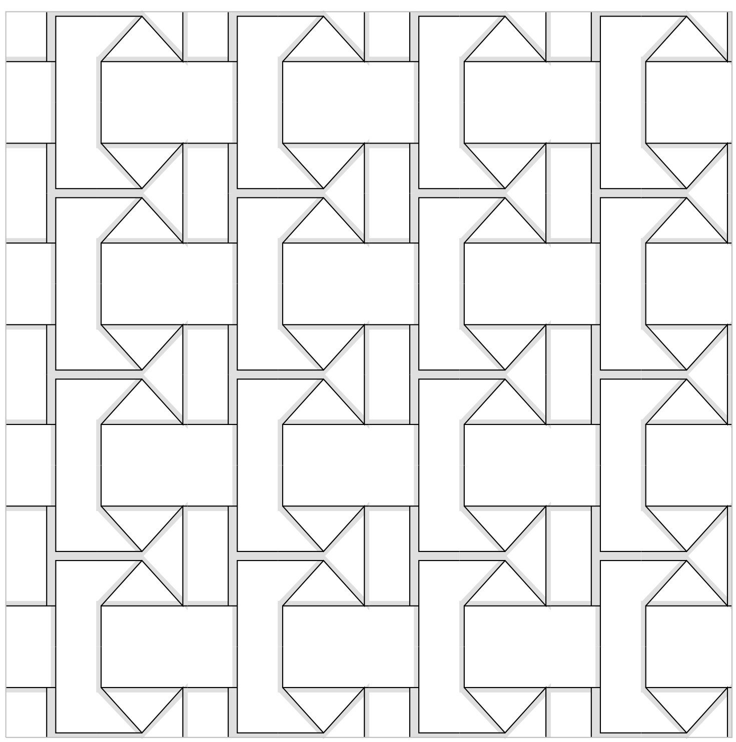

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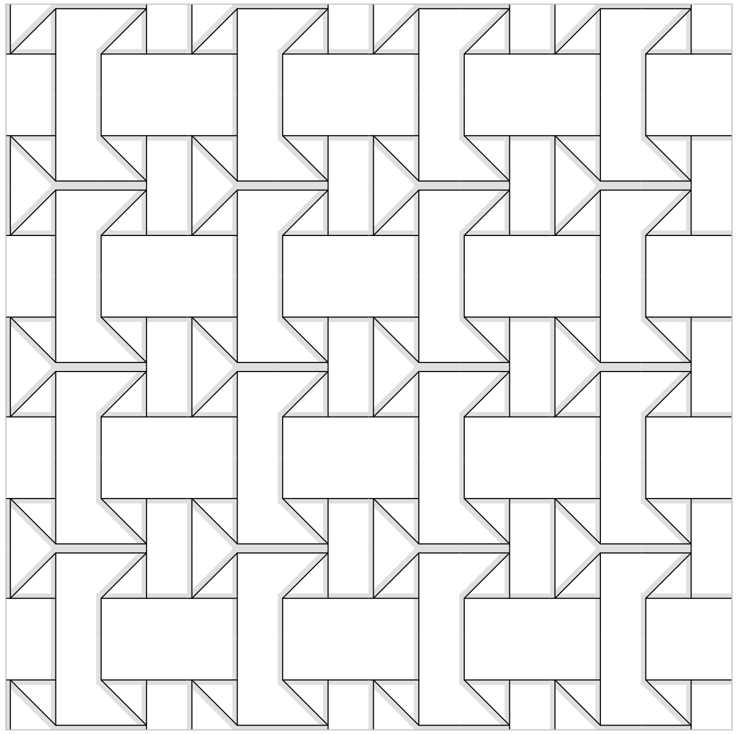

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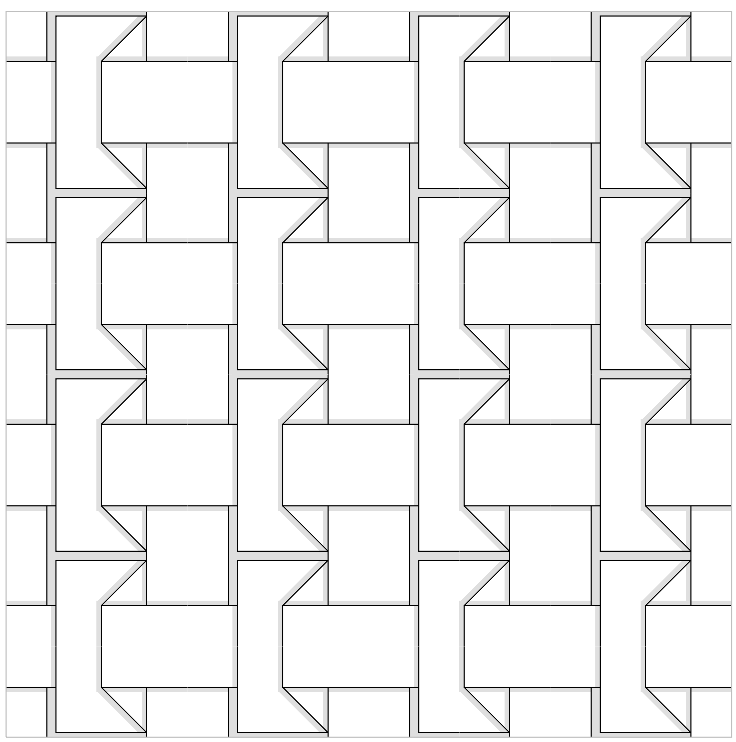

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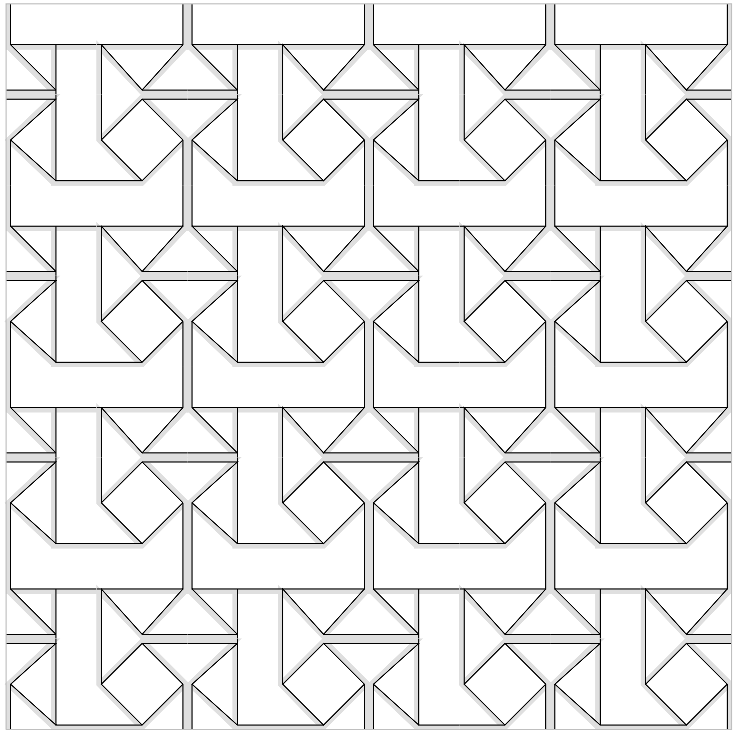

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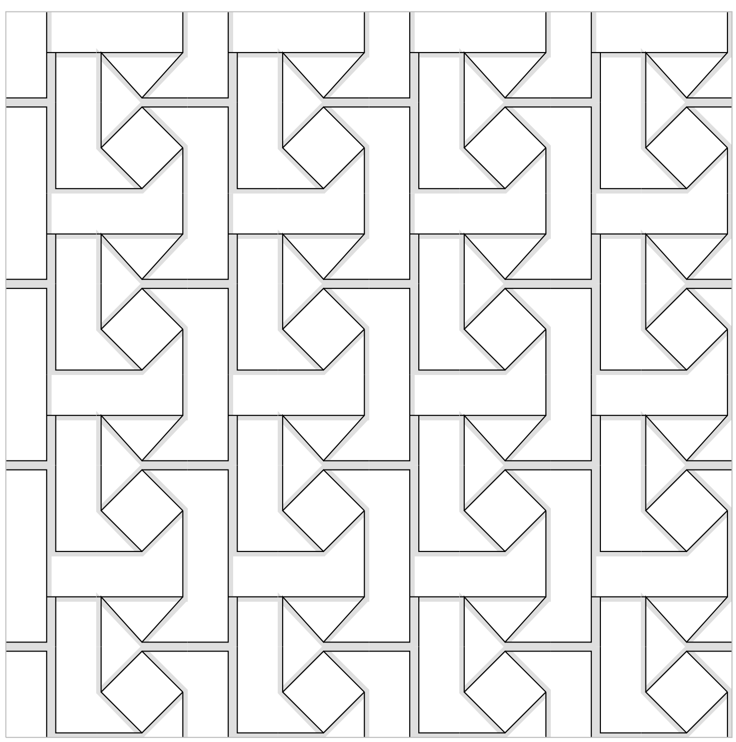

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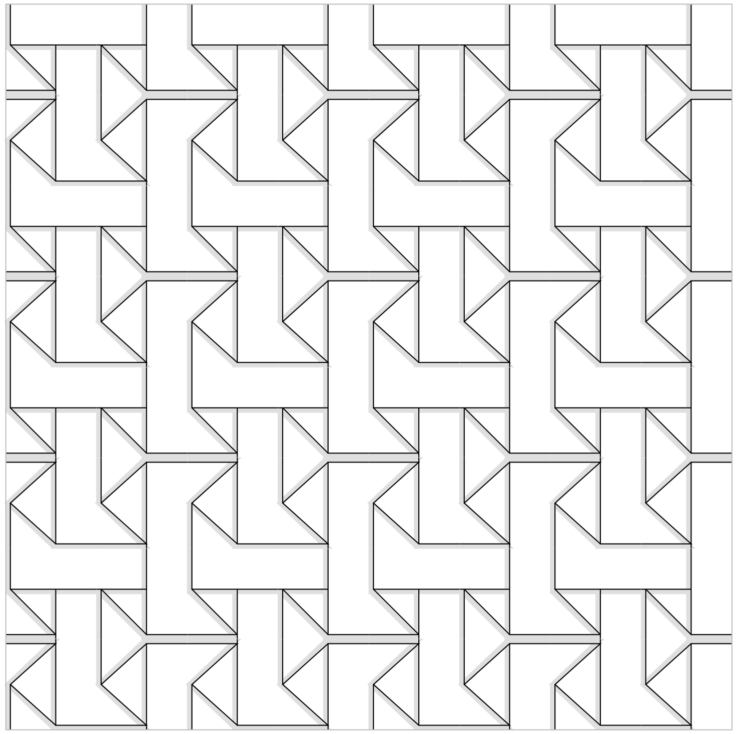

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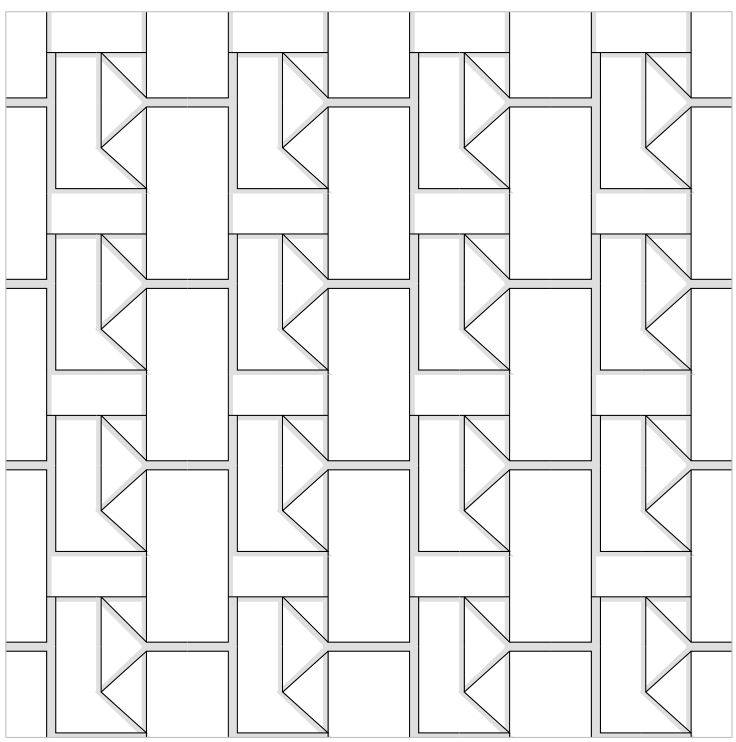

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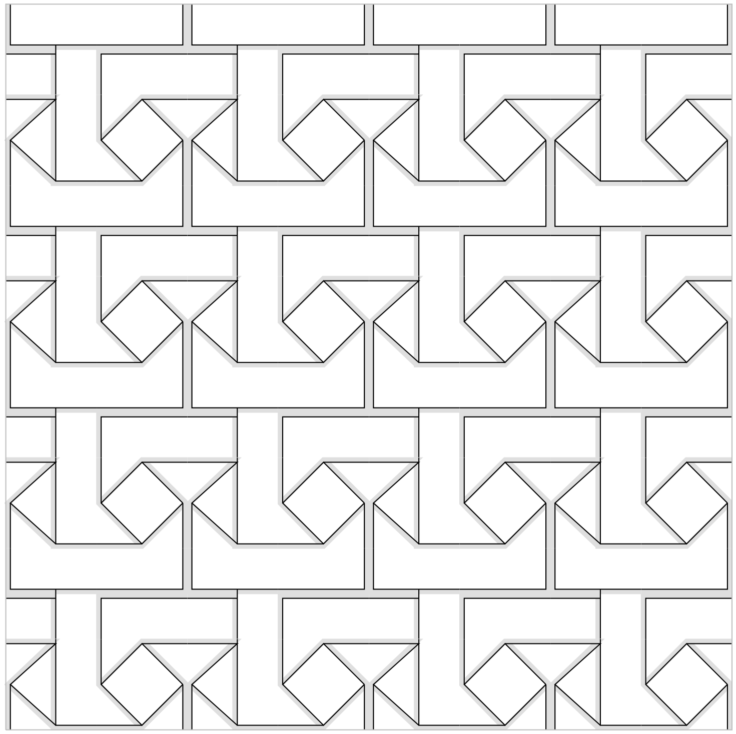

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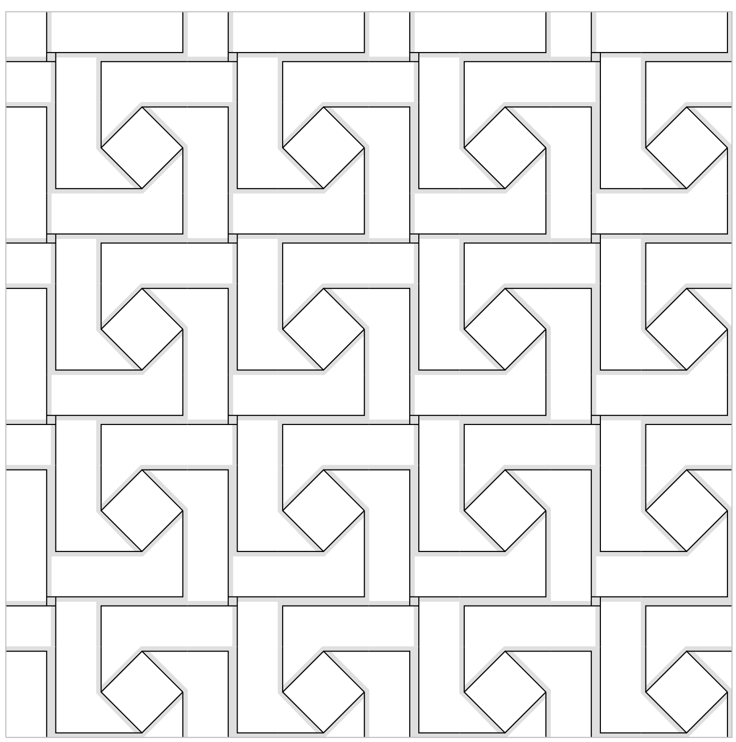

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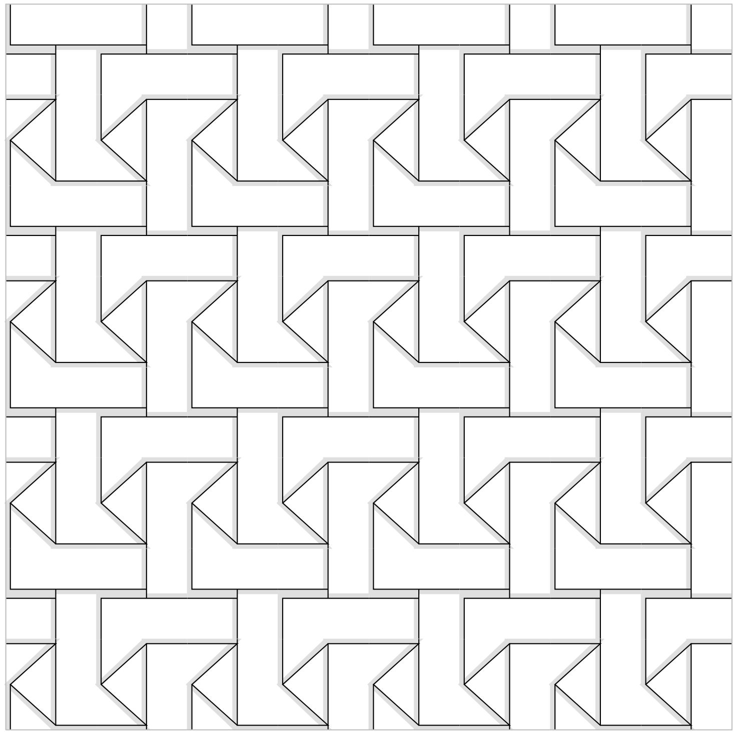

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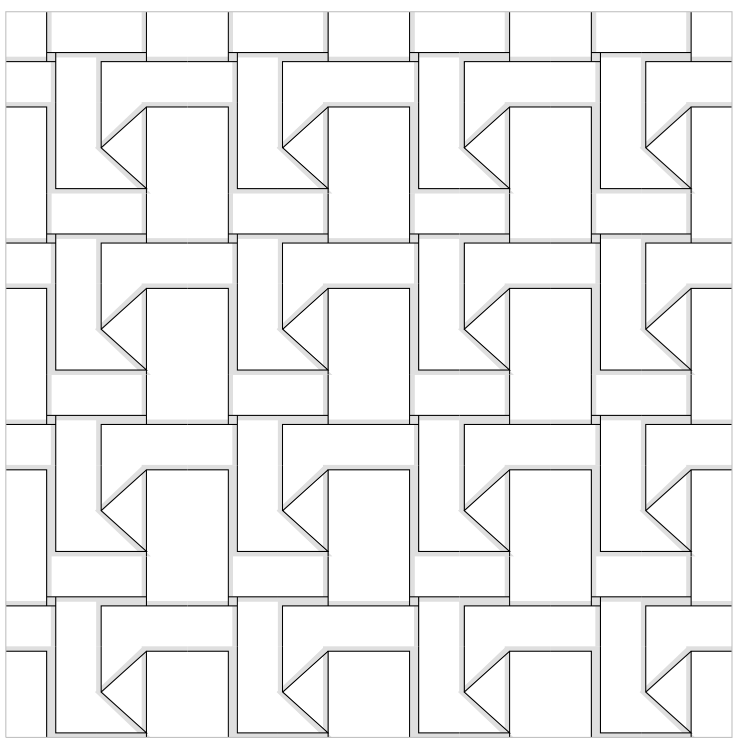

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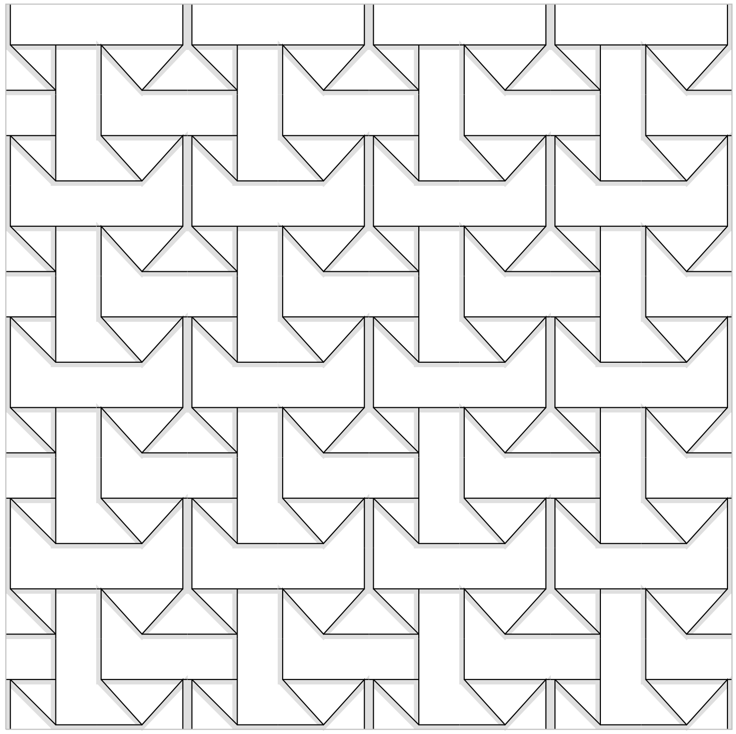

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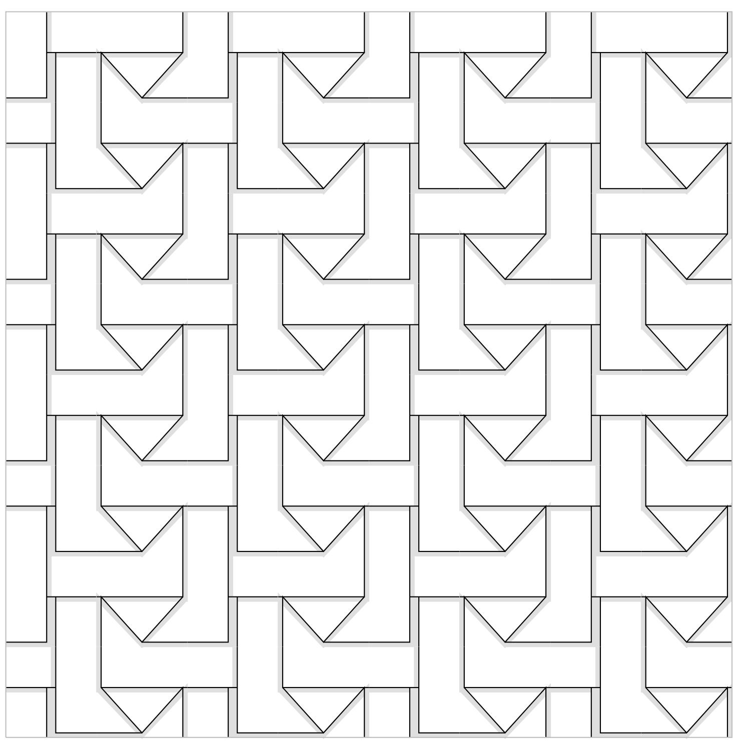

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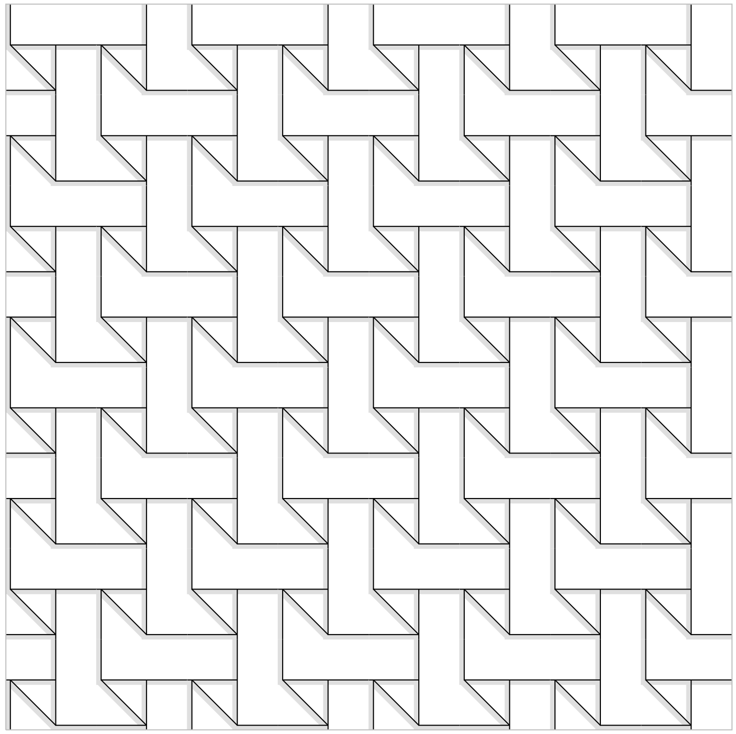

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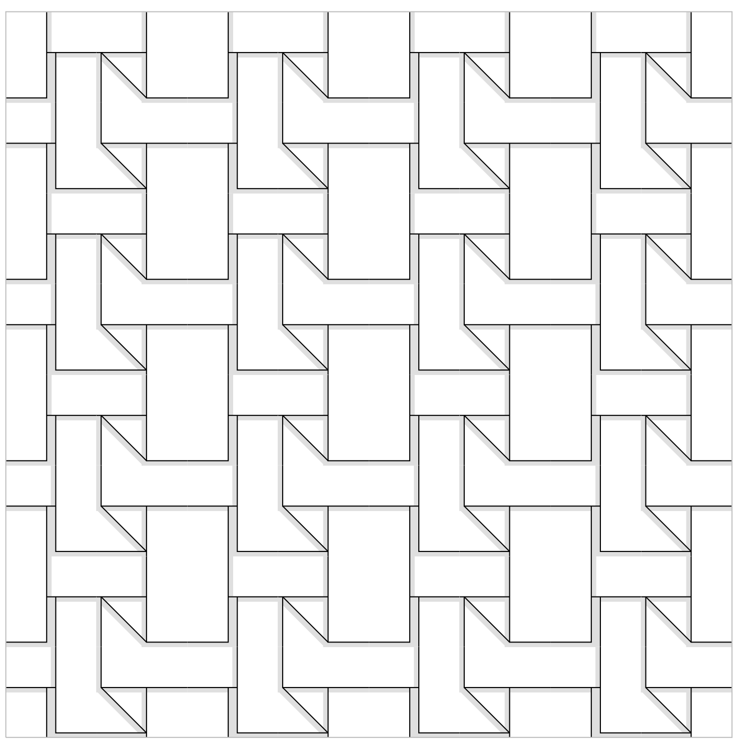

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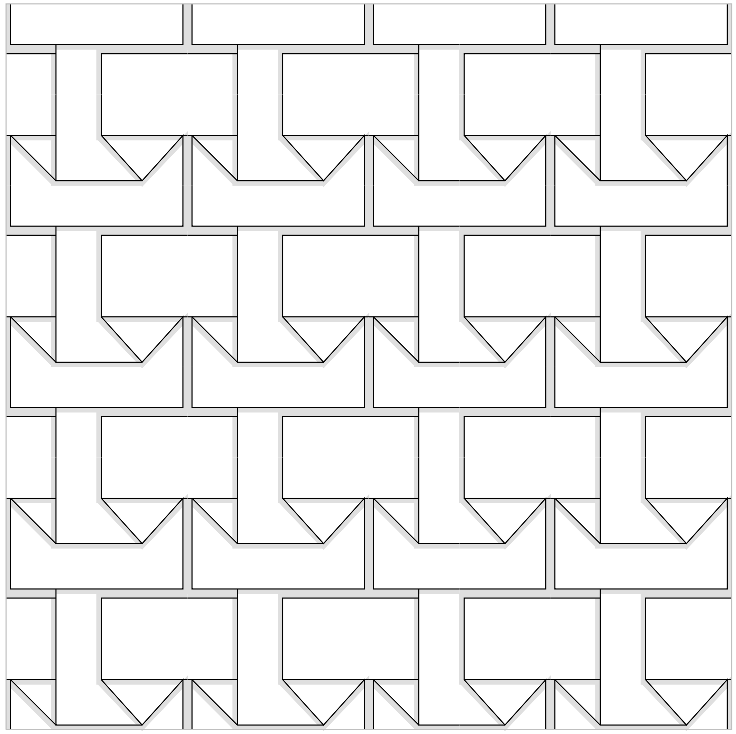

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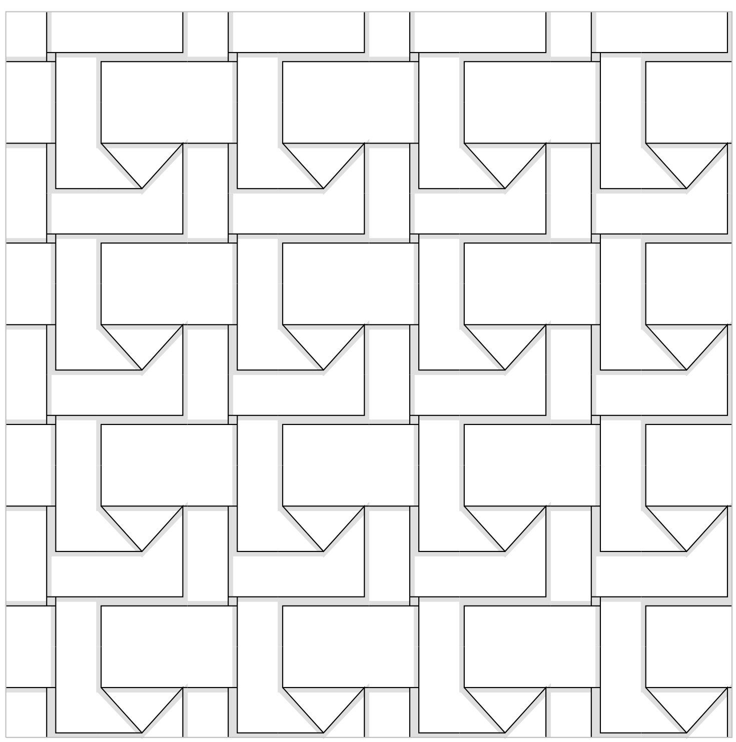

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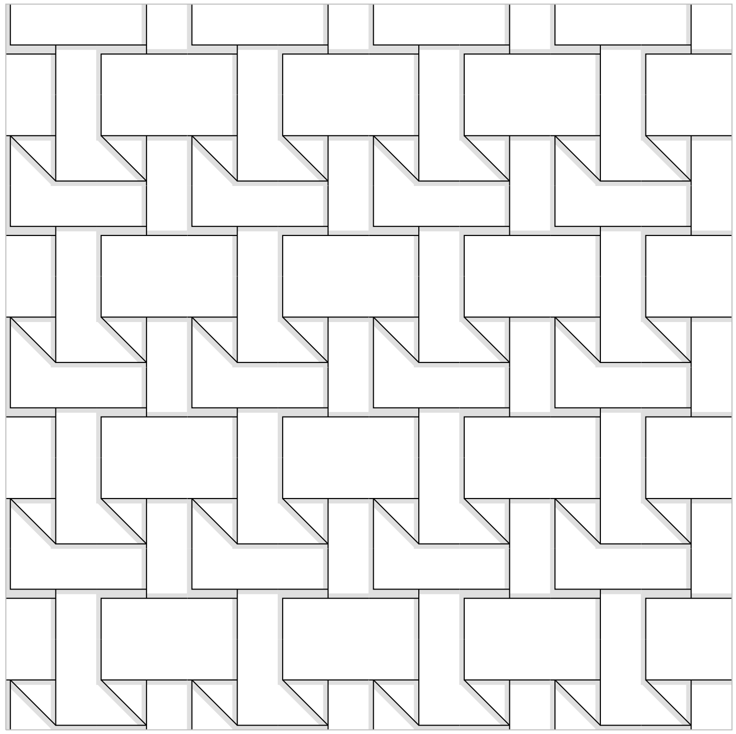

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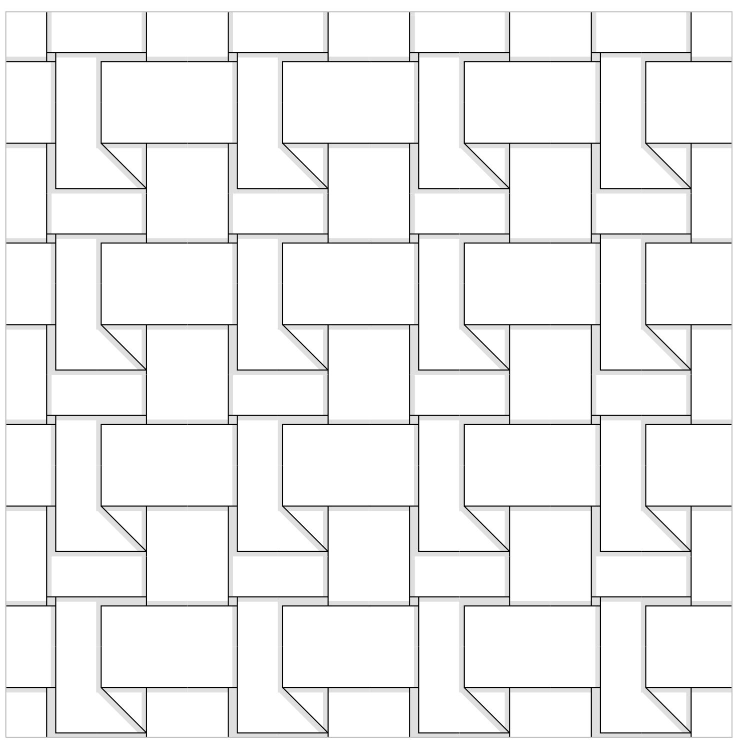

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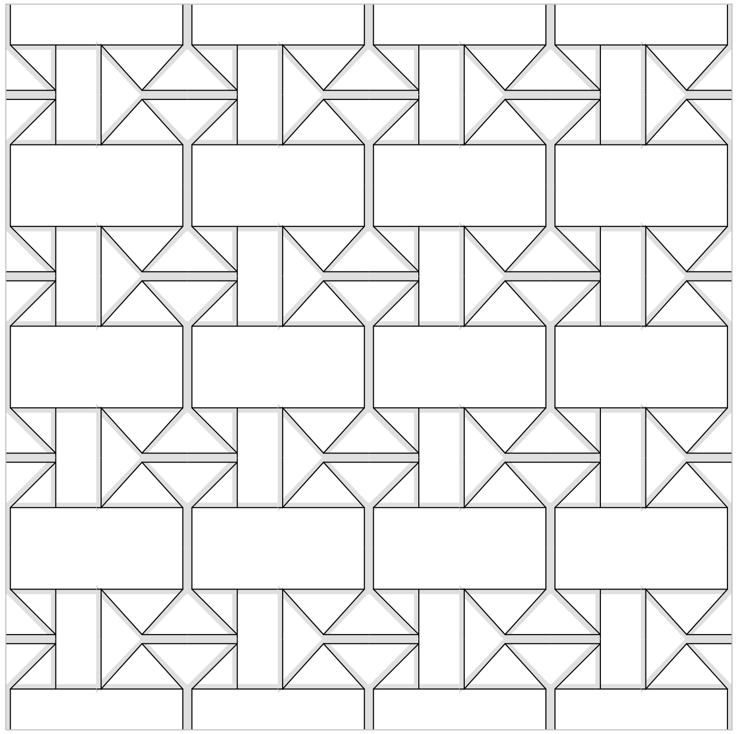

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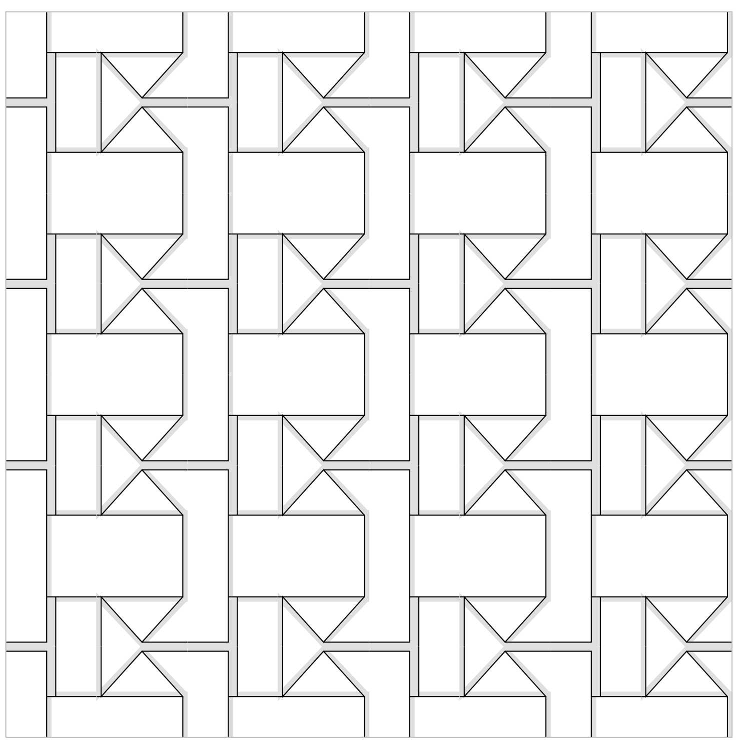

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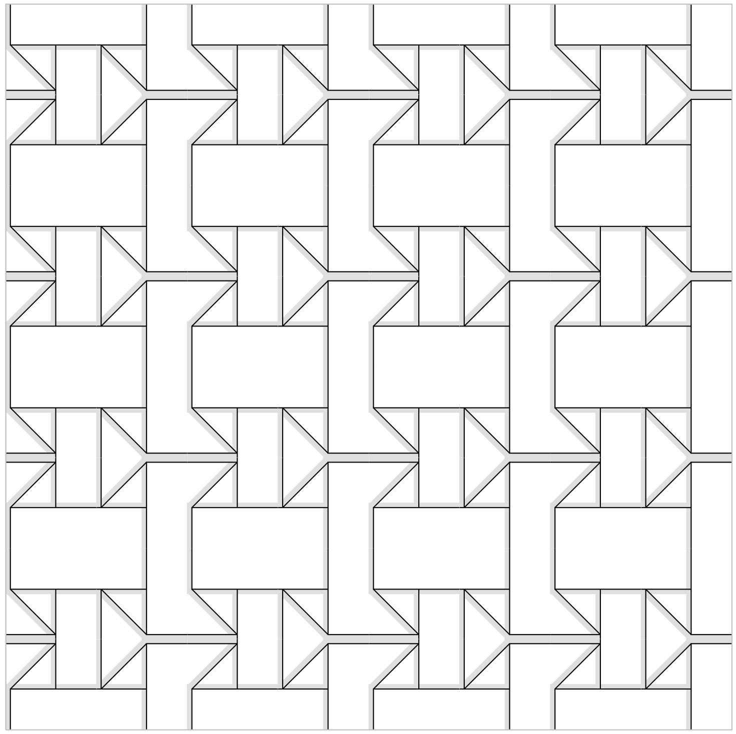

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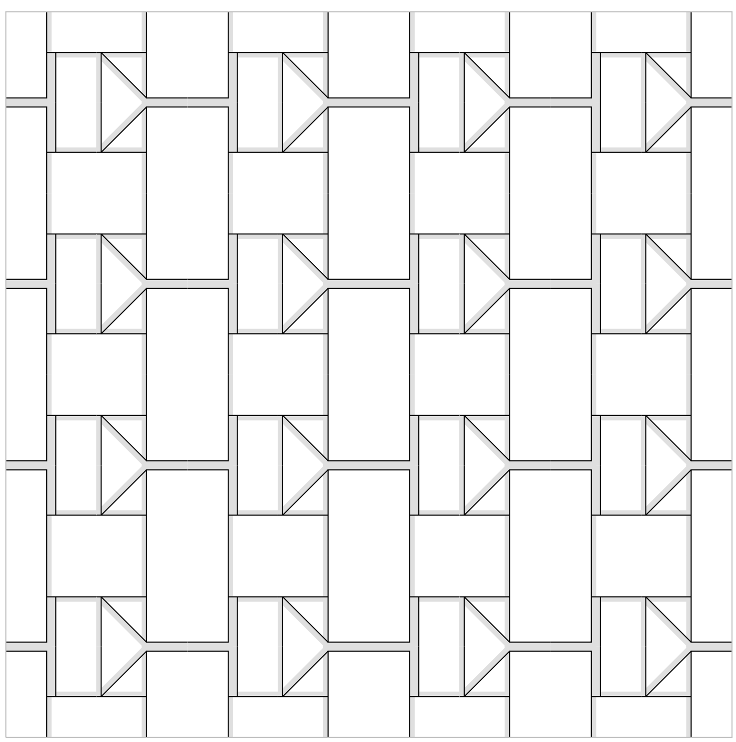

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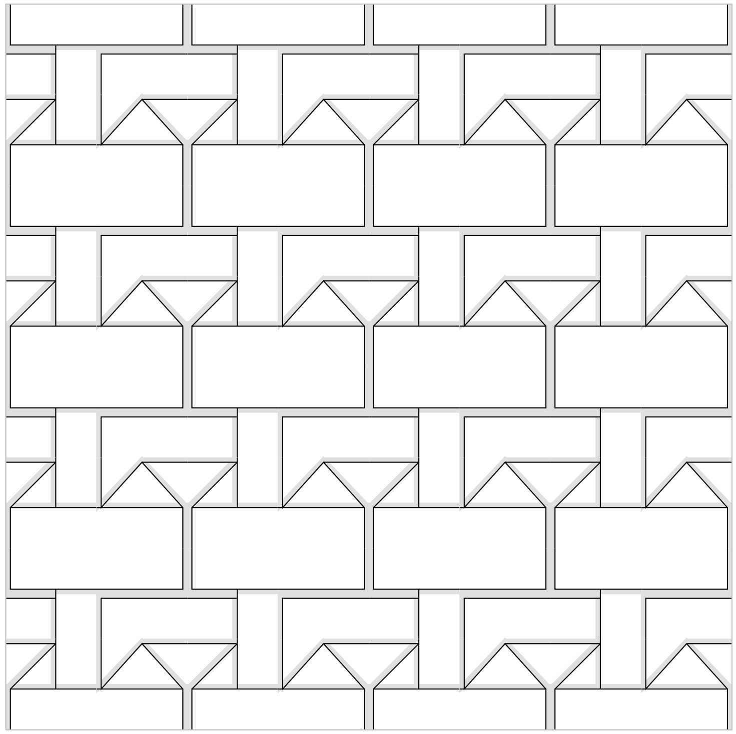

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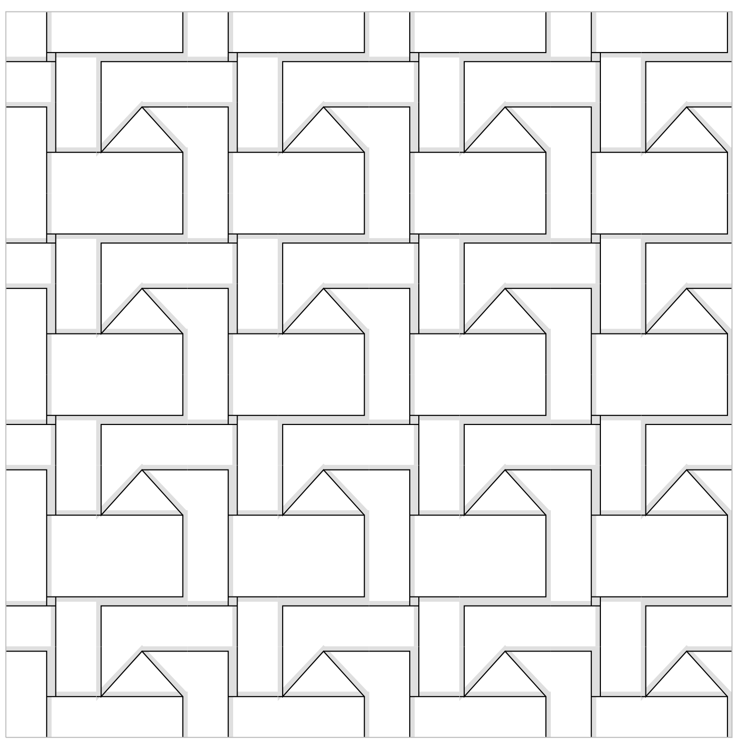

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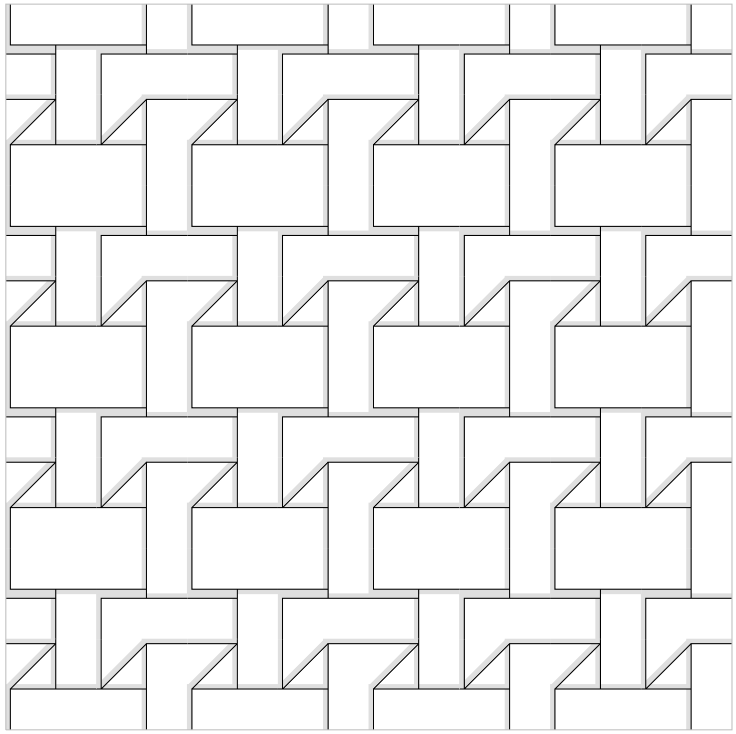

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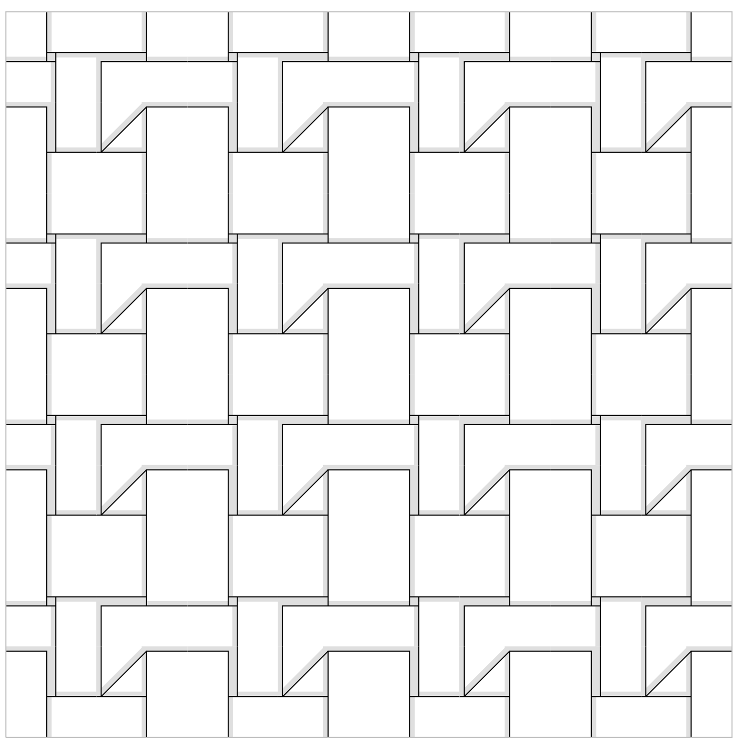

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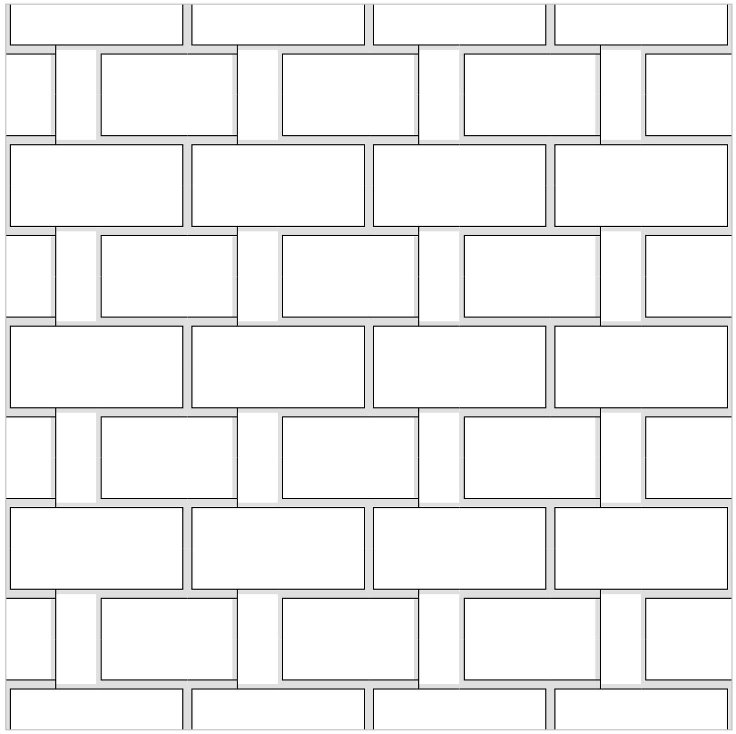

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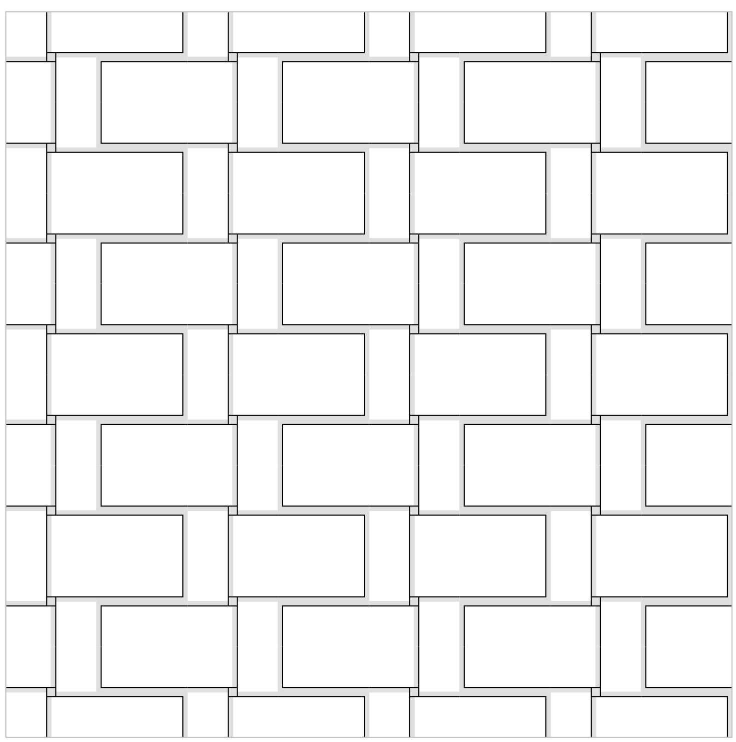

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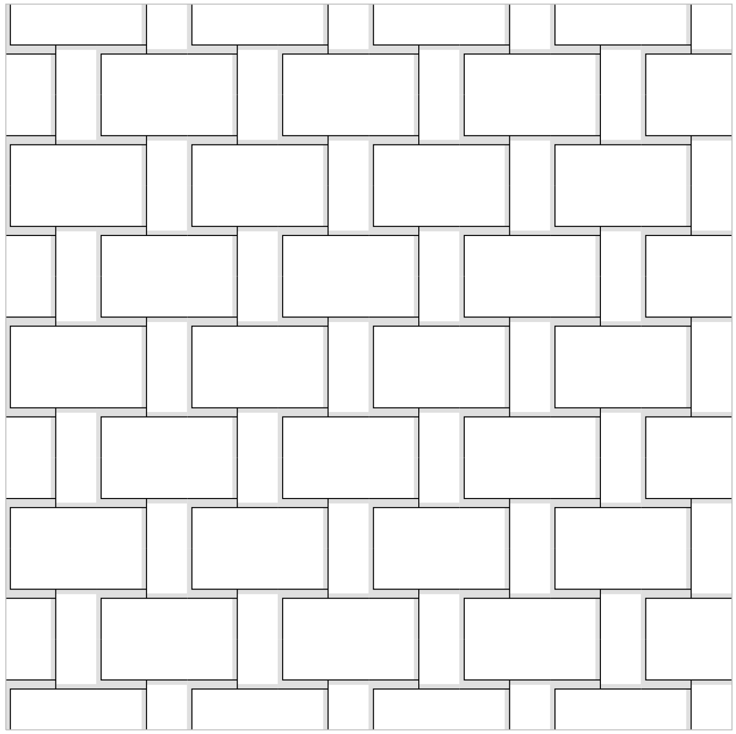

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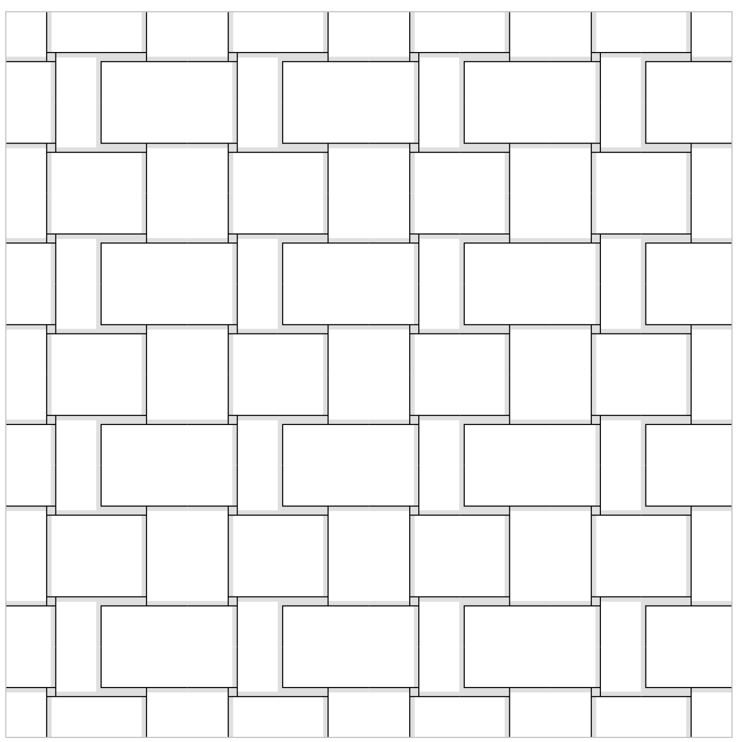

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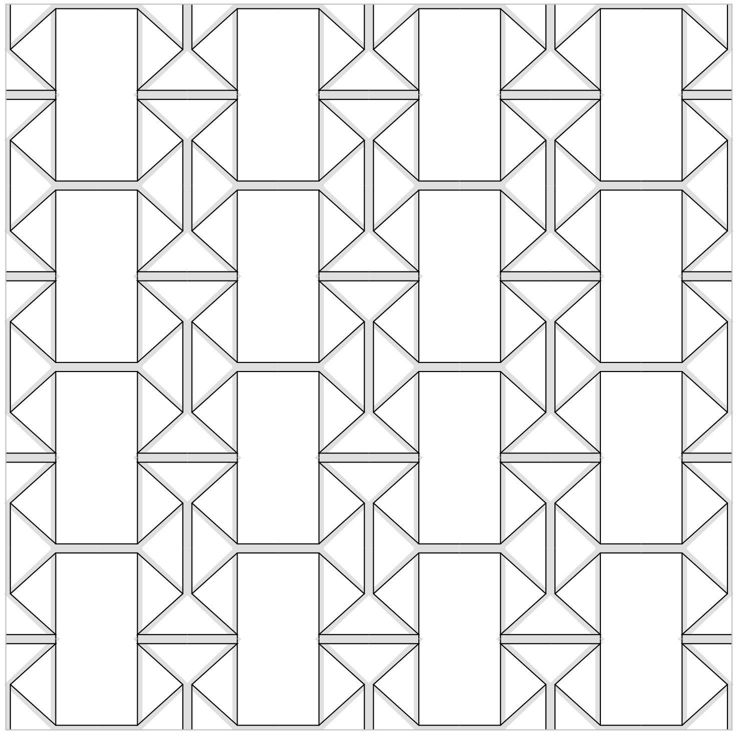

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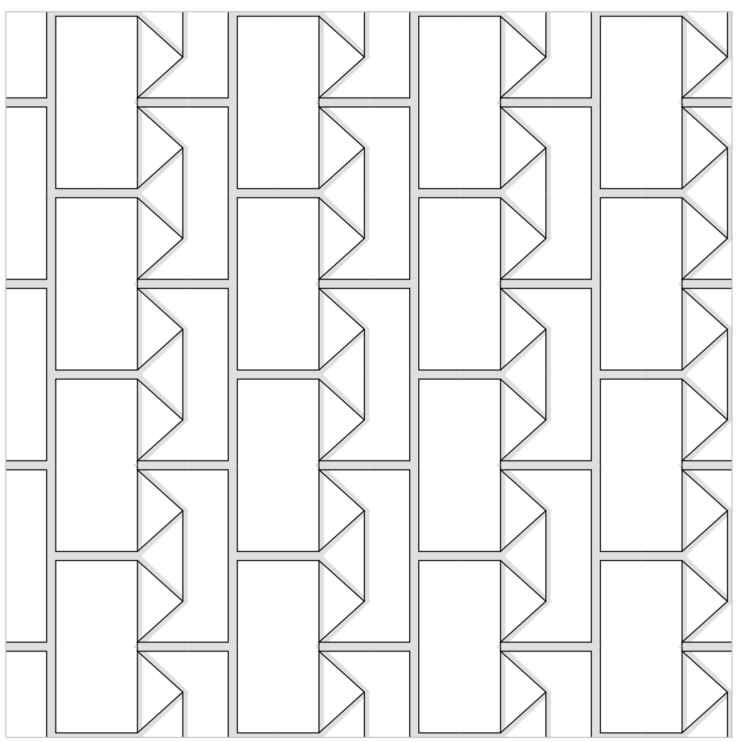

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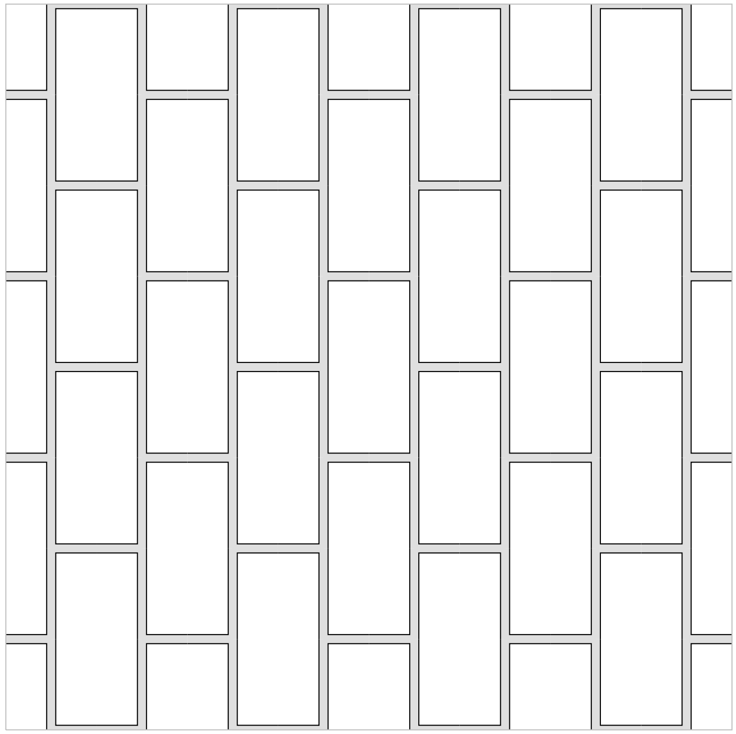

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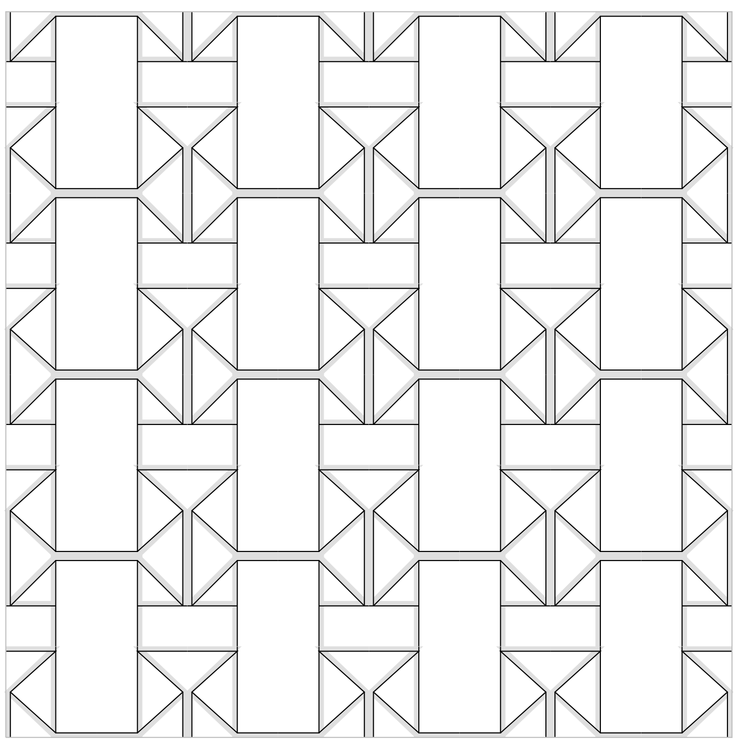

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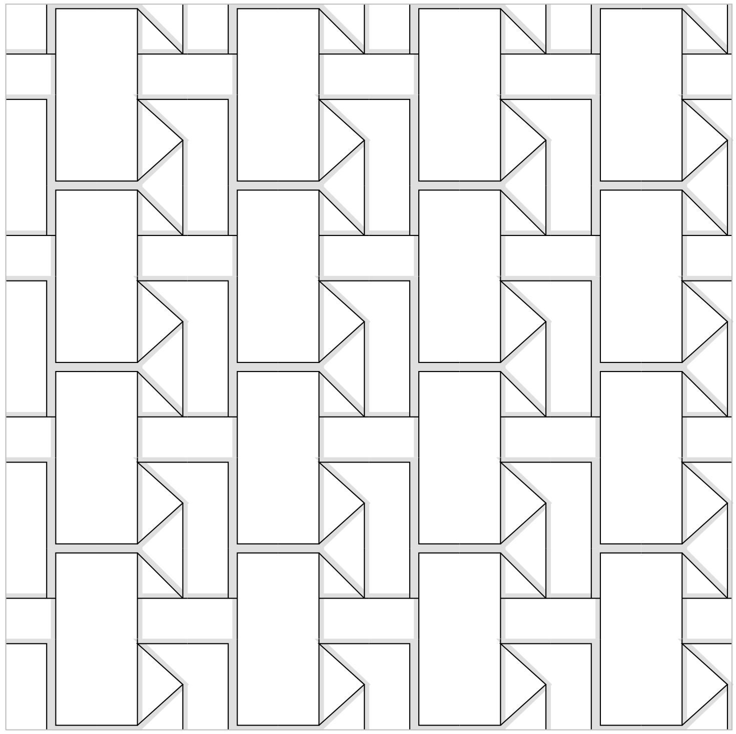

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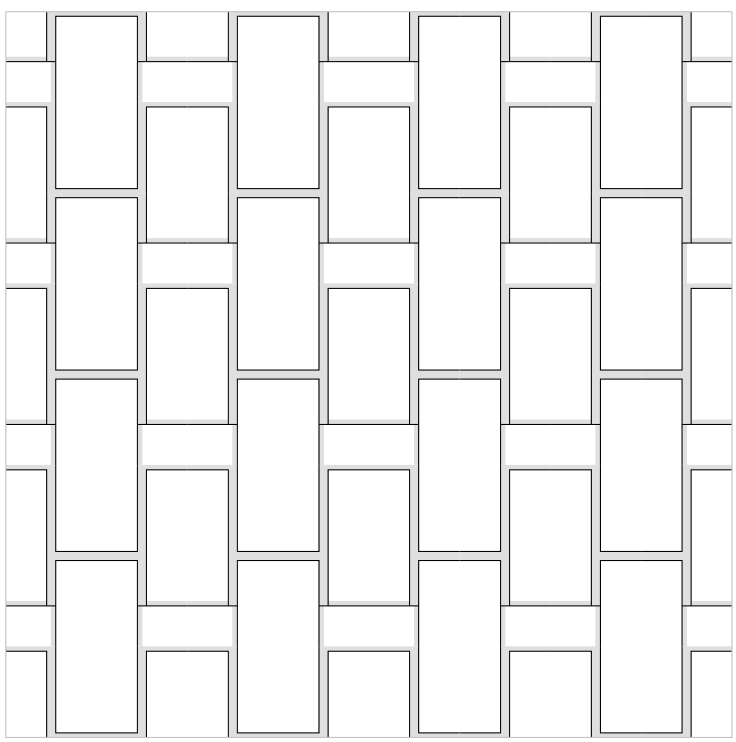

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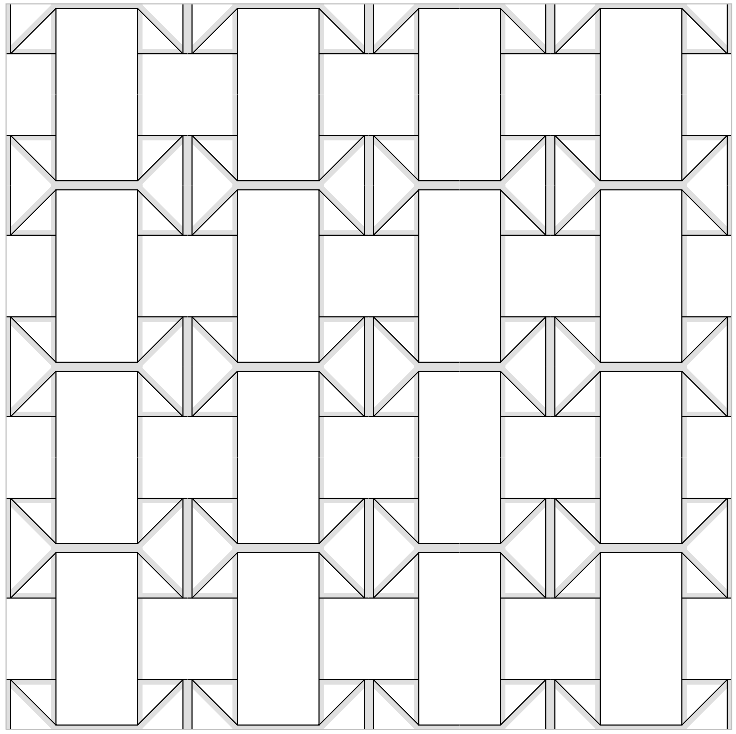

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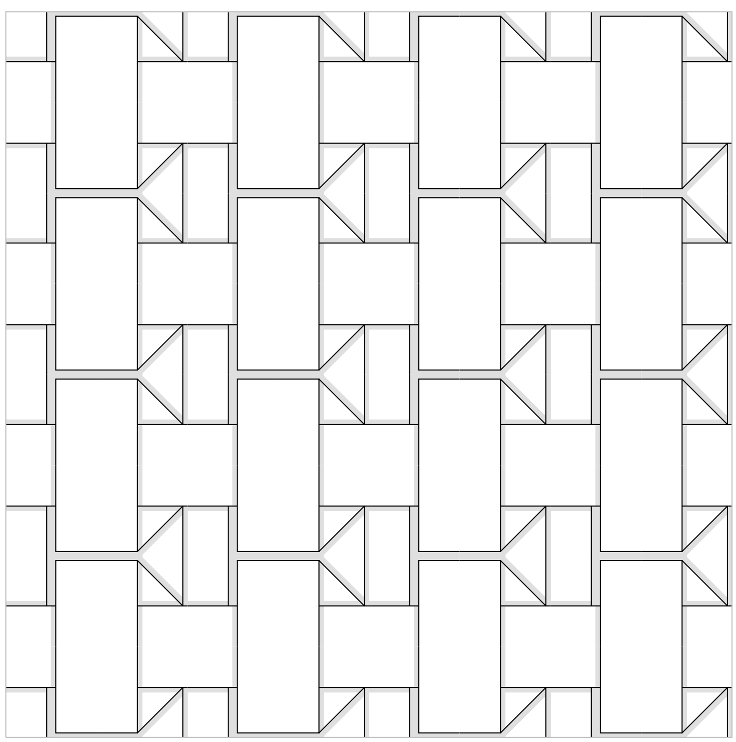

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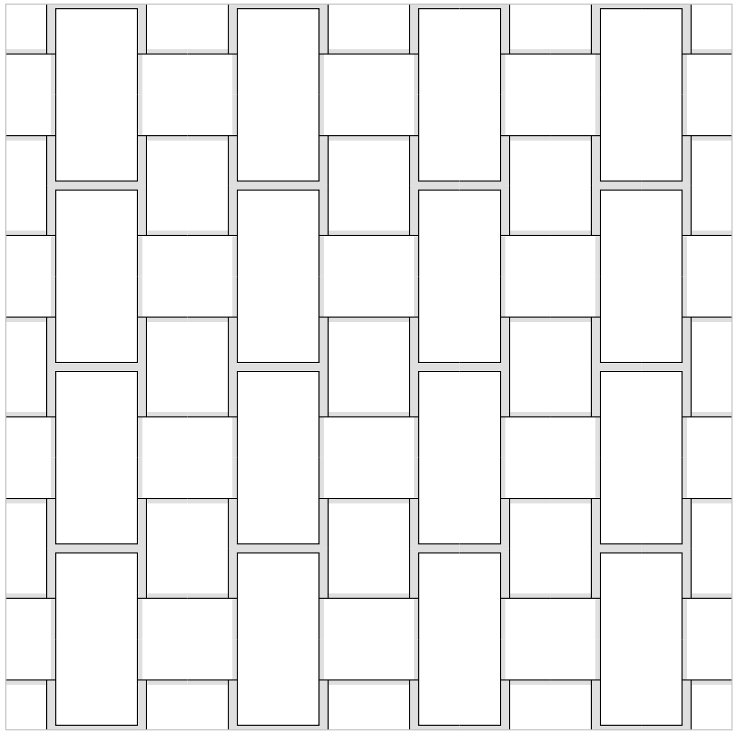

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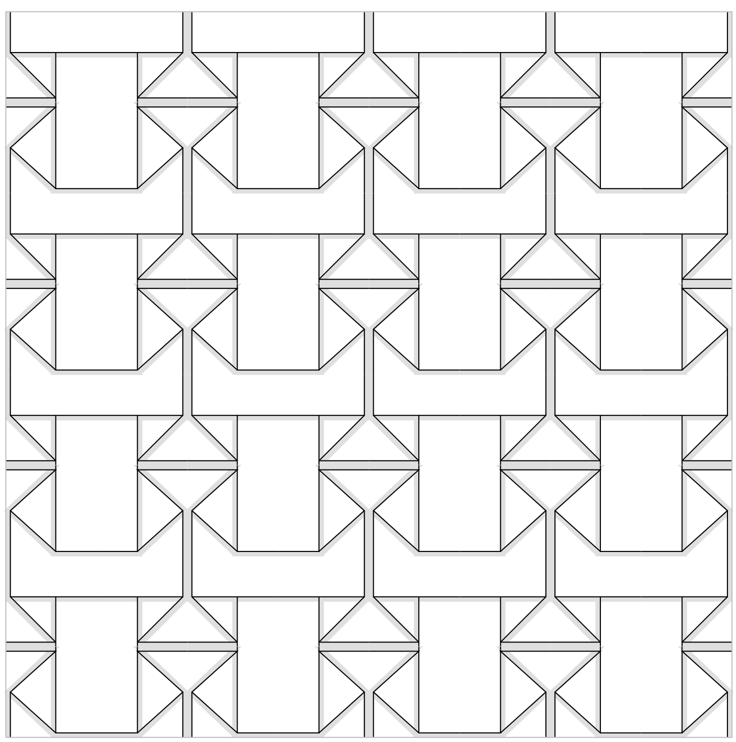

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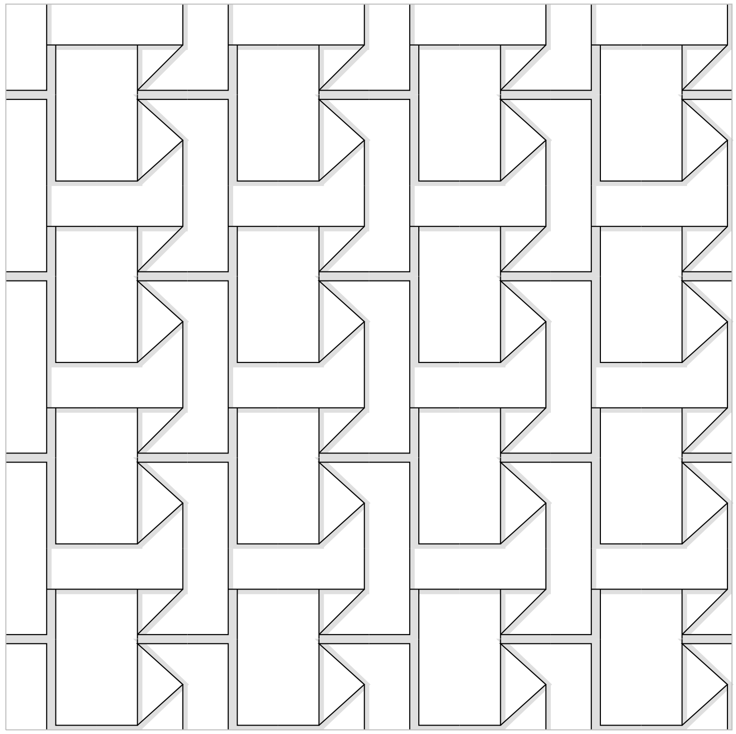

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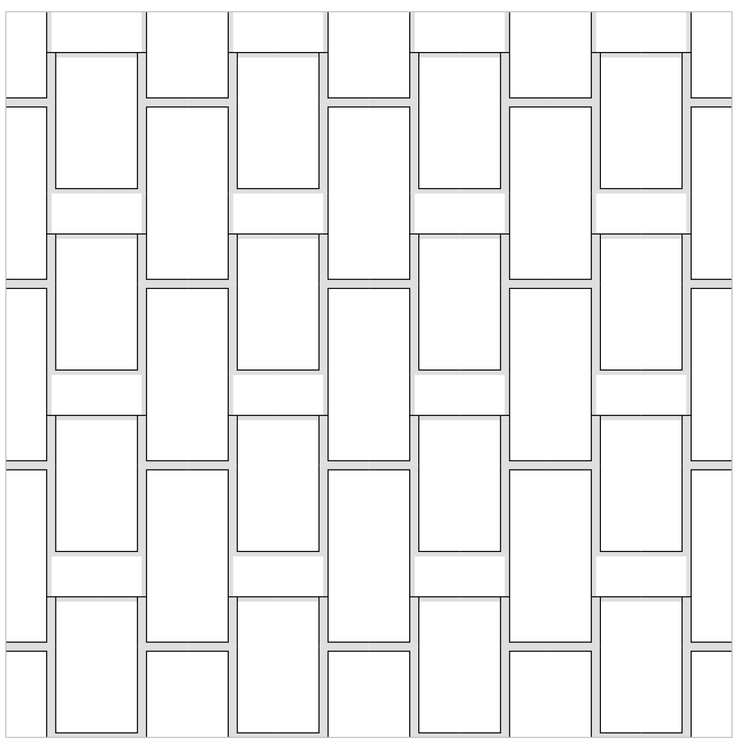

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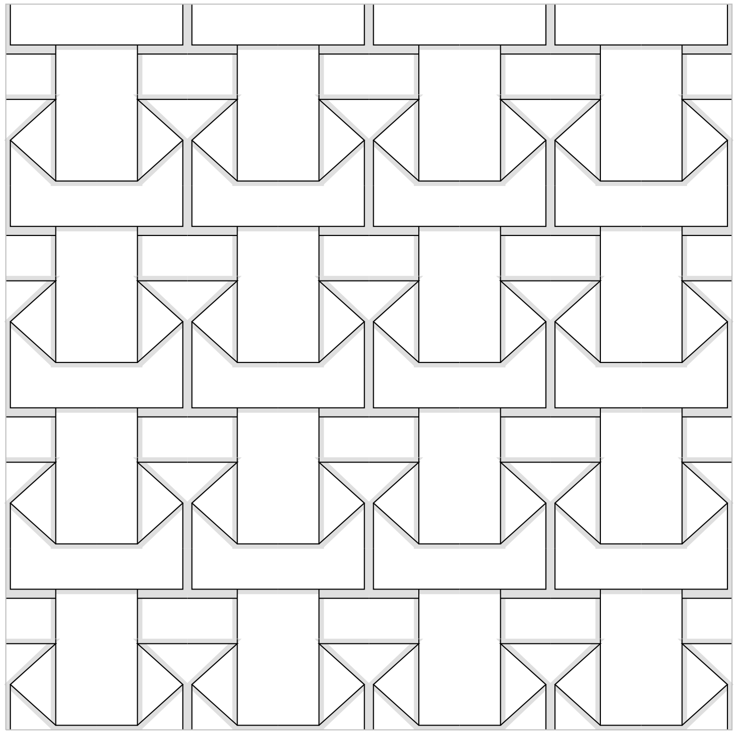

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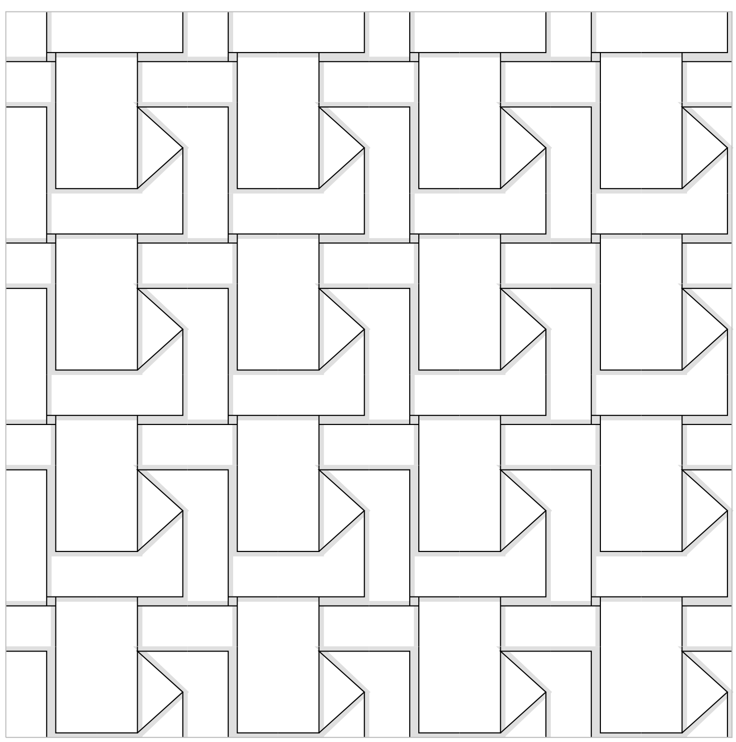

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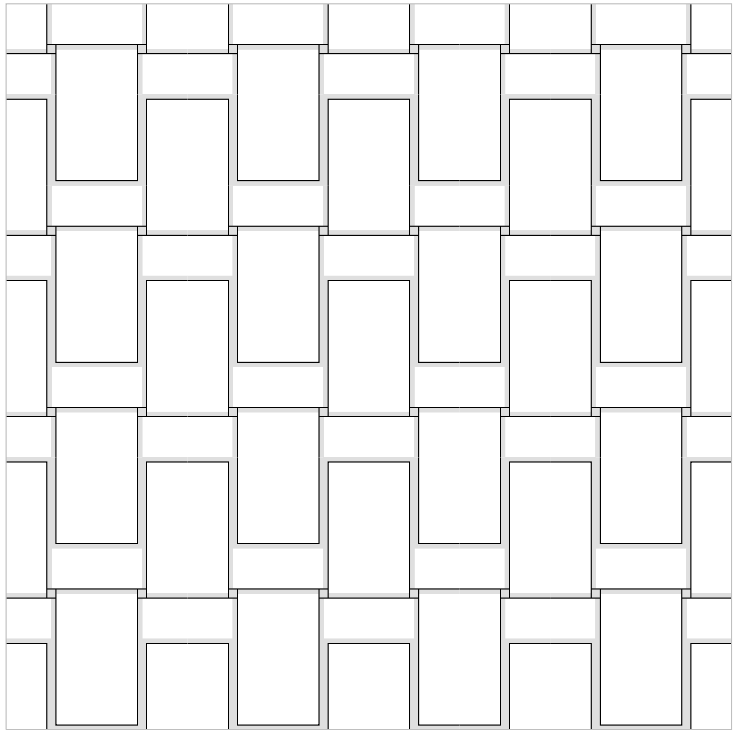

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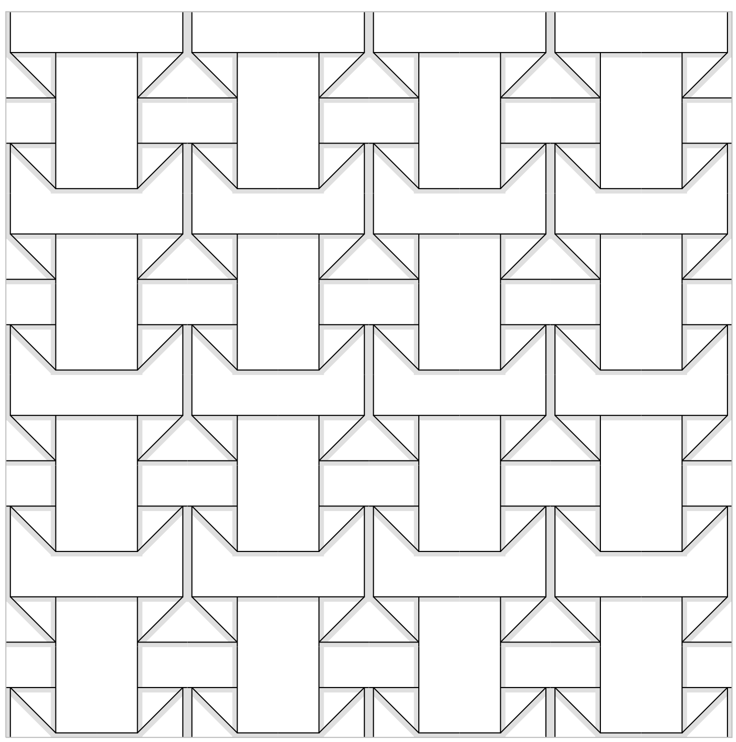

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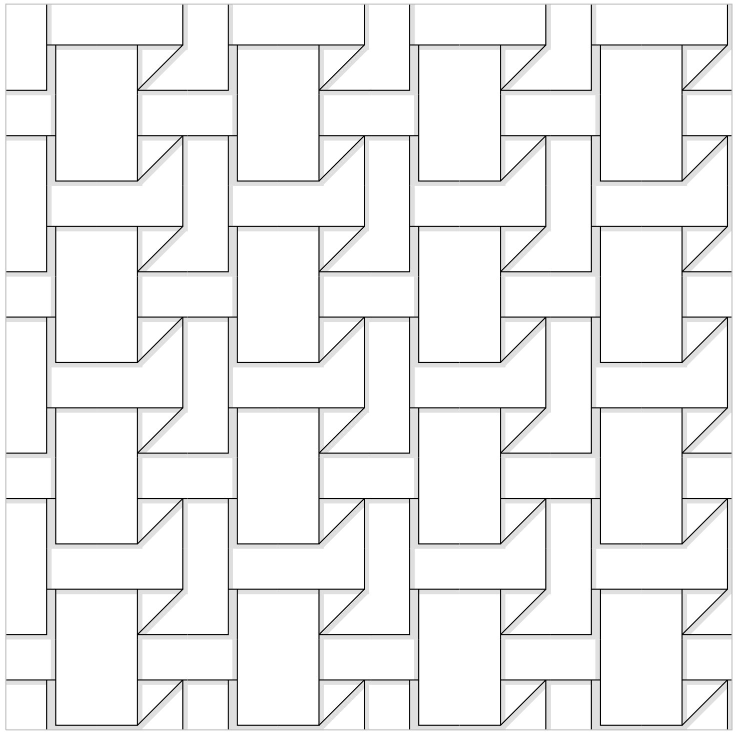

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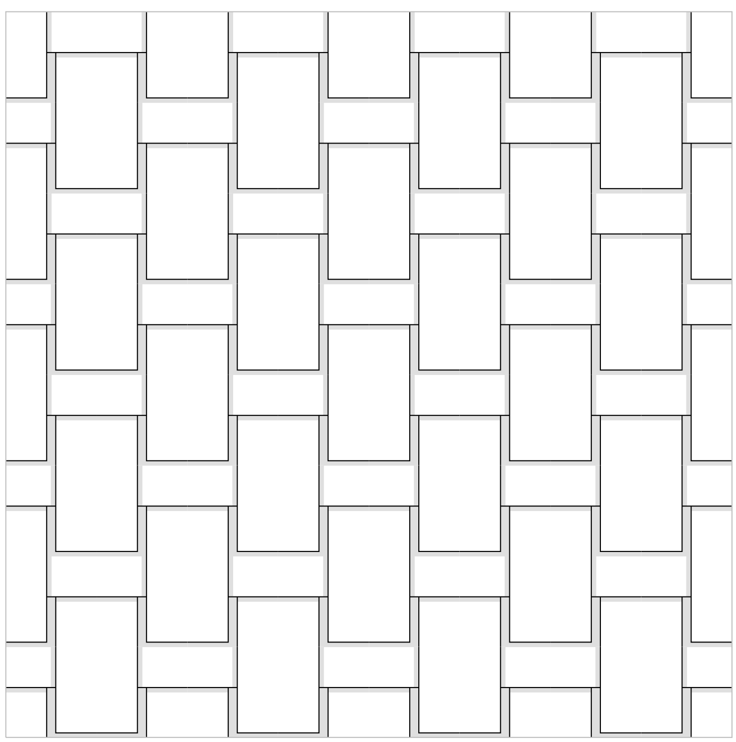

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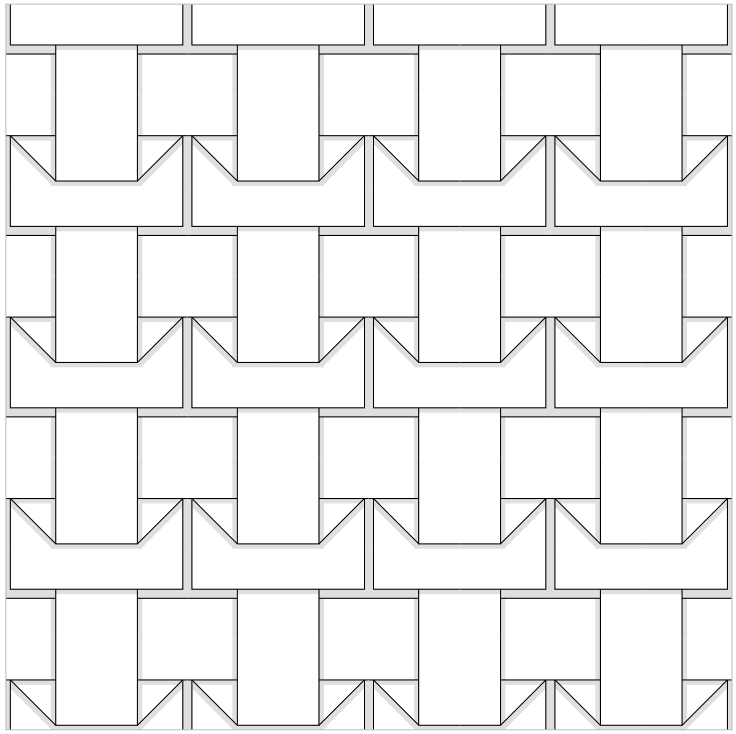

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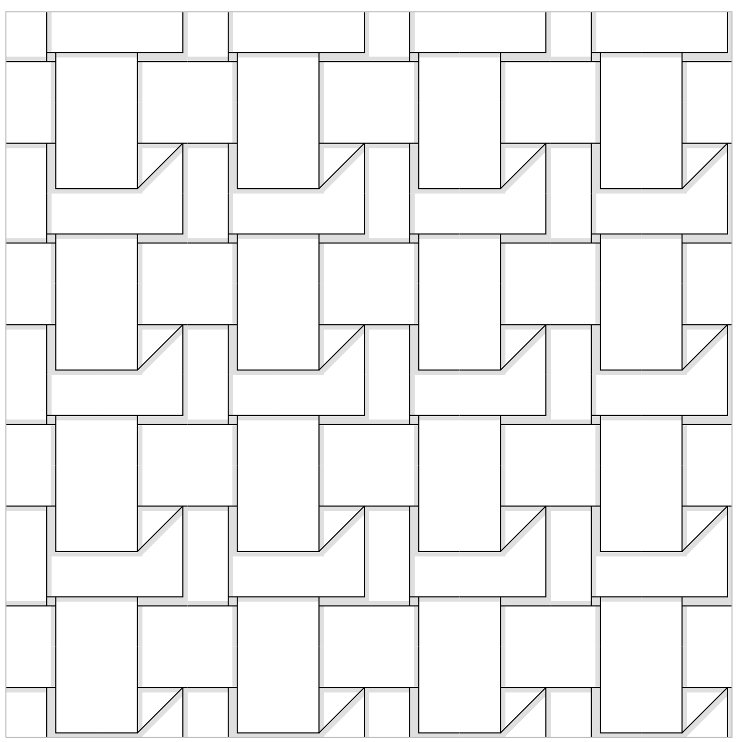

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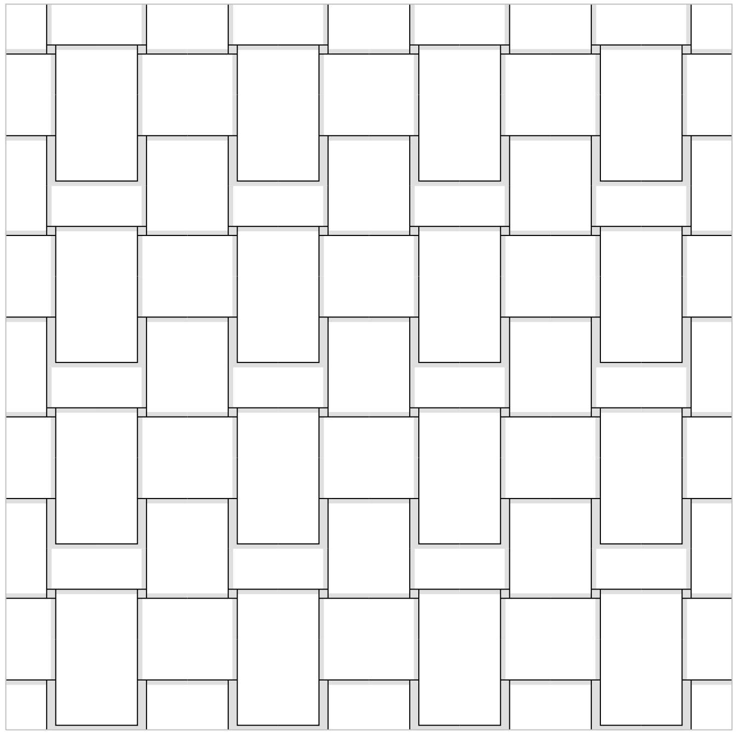


FIGURE 106. BBBBBABB

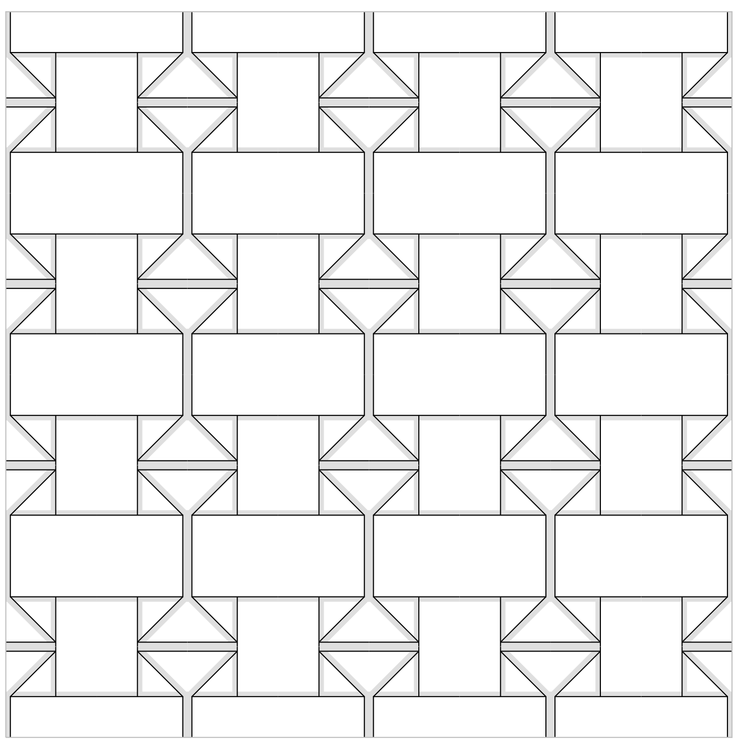
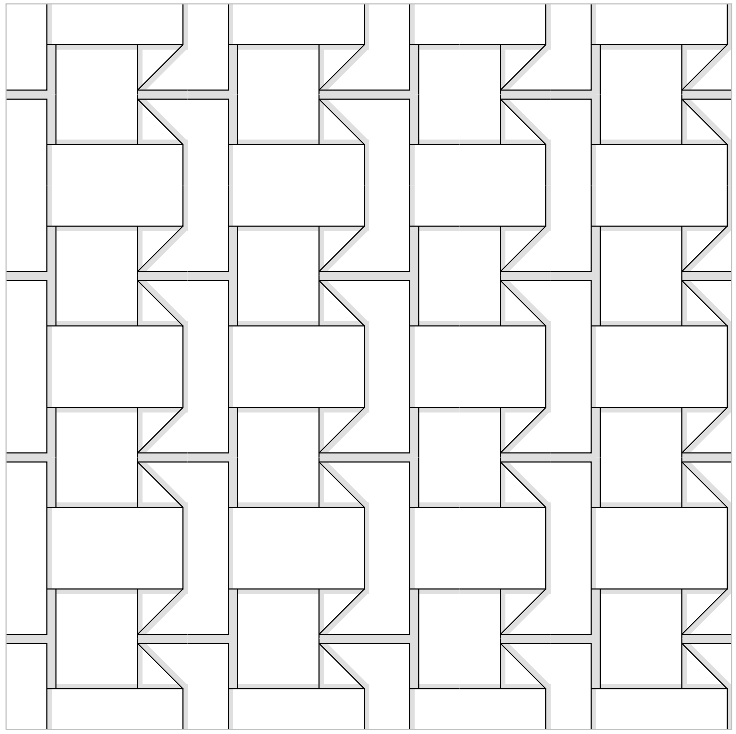


FIGURE 107. AAAABBBB



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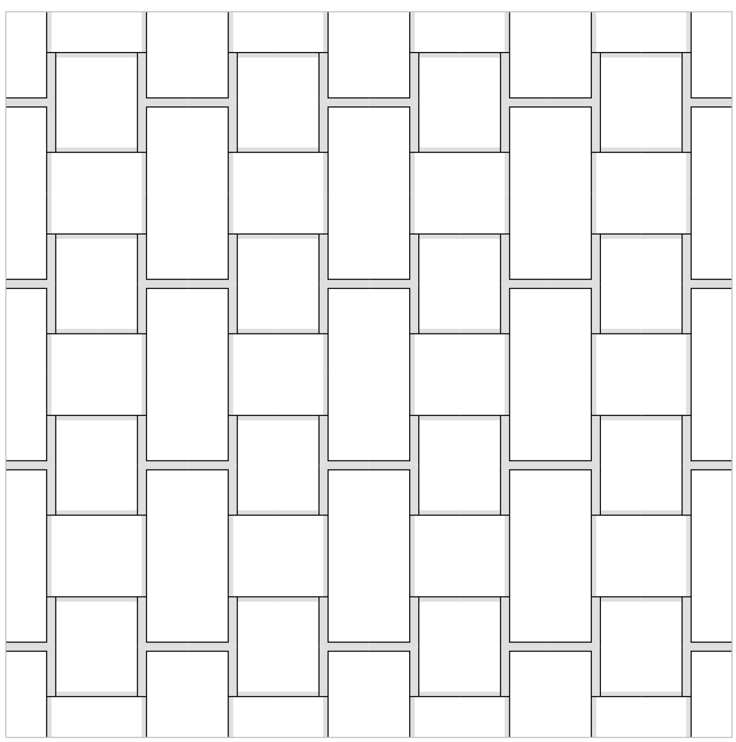

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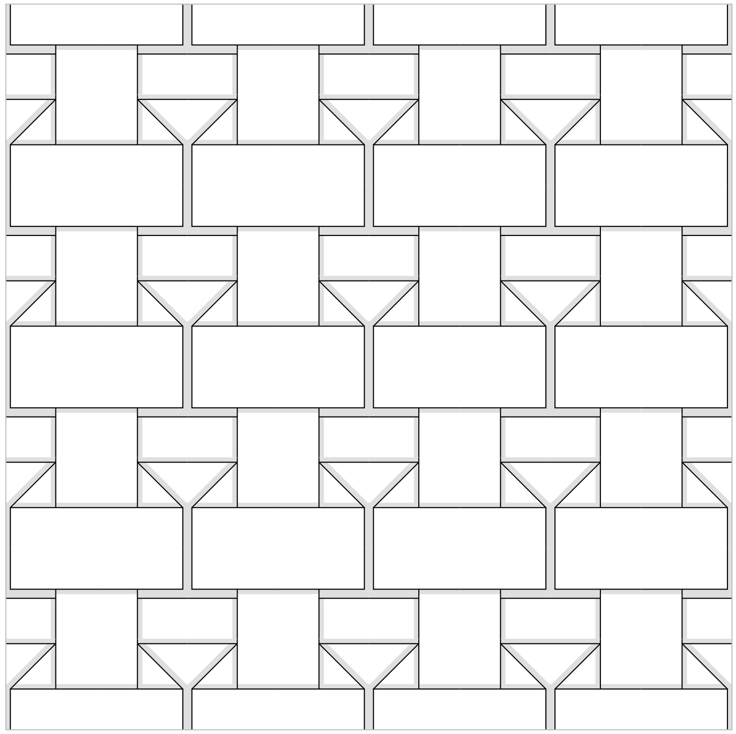

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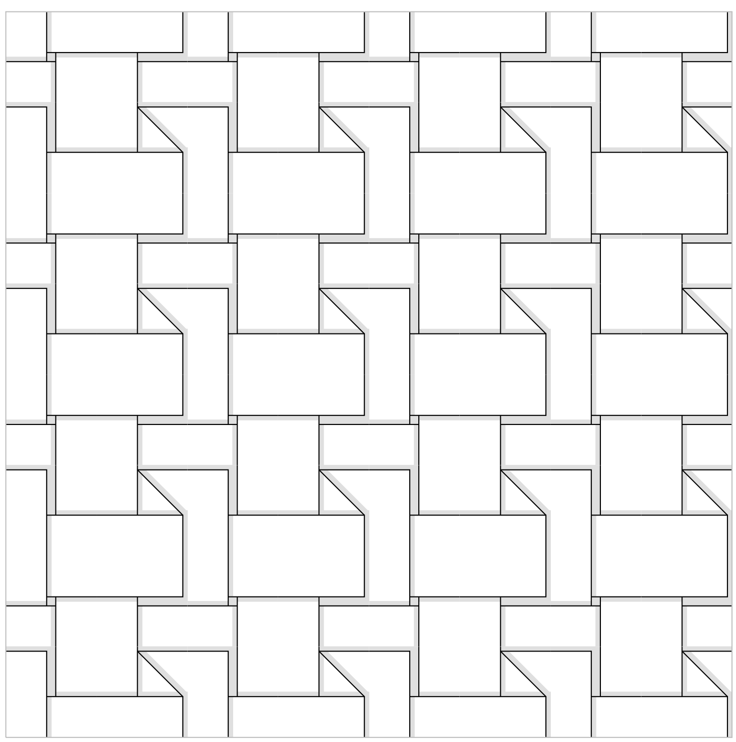

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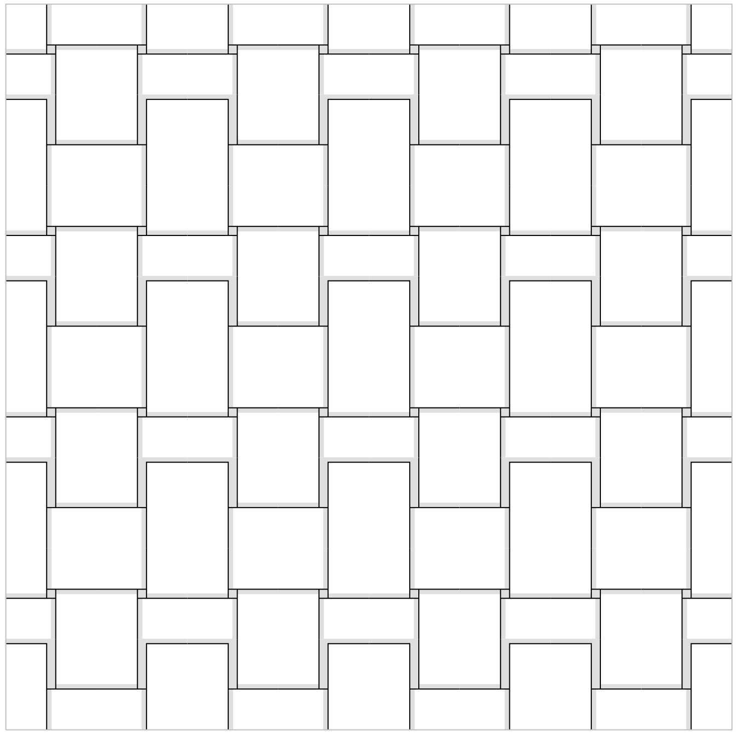

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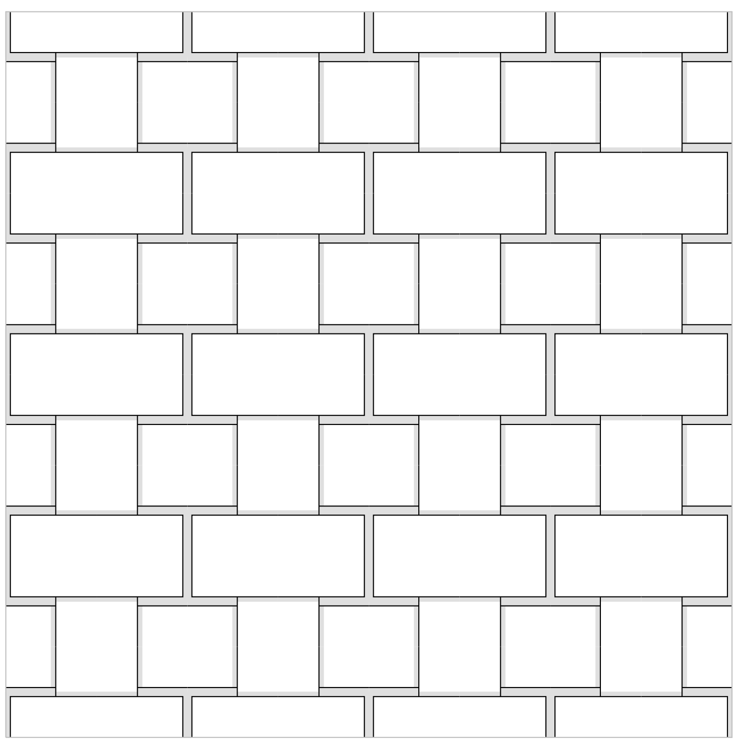

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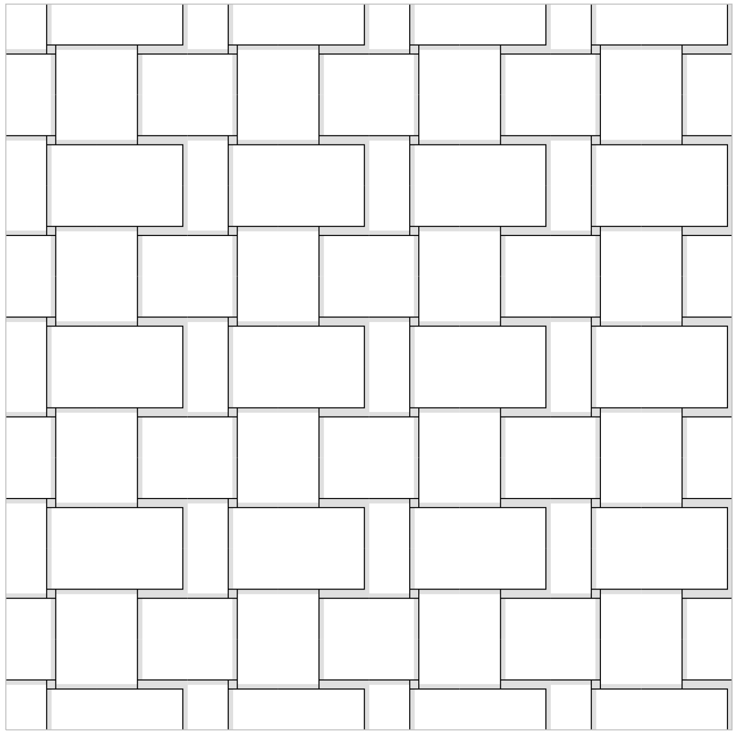

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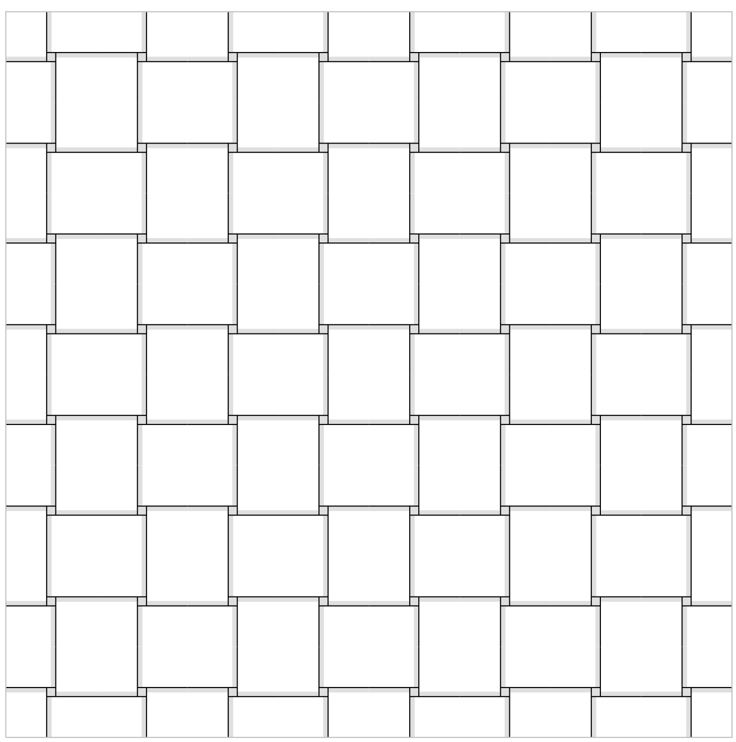

Figure 115. BBBBBBBB

references. [M] Momotani 1984 British Origami Society Convention Book

[V] Flat Herring bone origami tessellation, April 2019 `http://www.mathamaze.co.uk/origami/origamipdf/herringbone.pdf`